HW 2 (Due Sep 21, 2017) **Name:**

Problem 1.

Suppose that $Y_1, Y_2, ..., Y_n$ is an iid $\mathcal{N}(\mu, \sigma_0^2)$ sample, where $\sigma_0^2 = 9$. We would like to test

$$H_0 : \mu = 10$$
$$\text{versus}$$
$$H_a : \mu > 10.$$

We will use a rejection region of the form $\mathrm{RR} = \{\overline{y} : \overline{y} > k\}$. Find the values of n and k that provide a Type I Error probability of $\alpha = 0.02$ and a Type II Error probability of $\beta = 0.10$ when $\mu = 12$.

Problem 2.

Suppose that $Y_1, Y_2, ..., Y_n$ is an iid sample of size n from $f_Y(y;\theta)$, where

$$f_Y(y;\theta) = \begin{cases} \theta y^{\theta-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

We are interested in testing $H_0 : \theta = 1$ versus $H_a : \theta < 1$. To perform the test, suppose we use the test statistic

$$T = \prod_{i=1}^{n} Y_i$$

and the rejection region $\text{RR} = \{t : t < k\}$, where k is a constant.

(a) Show that T is a sufficient statistic for θ.

(b) Find an expression for k that makes RR have Type I Error probability equal to α.

Hint: Show that $U = -2\ln T \sim \chi^2(2n)$ when H_0 is true.

Problem 3.

Suppose that $Y_1, Y_2, ..., Y_n$ is an iid $\mathcal{N}(0, \sigma^2)$ sample, where $\sigma^2 > 0$ is unknown. We are interested in testing

$$\begin{gathered} H_0 : \sigma^2 = \sigma_0^2 \\ \text{versus} \\ H_a : \sigma^2 > \sigma_0^2, \end{gathered}$$

where σ_0^2 is known. To perform the test, suppose we use the test statistic

$$T = \sum_{i=1}^{n} Y_i^2$$

and the rejection region $\text{RR} = \{t : t > k\}$, where k is a constant.

(a) Show that T is a sufficient statistic for σ^2.

(b) Determine the value of k that makes RR have Type I Error probability equal to α.

(c) Using your answer from part (b), find an expression for $\beta(\sigma_a^2)$, for $\sigma_a^2 \geq \sigma_0^2$. Show that $\beta(\sigma_a^2)$ is a decreasing function of σ_a^2.

Problem 4.

Suppose that $Y_1, Y_2, ..., Y_n$ is an iid sample from a Poisson distribution with mean θ. Suppose that we would like to test, at level α,

$$H_0 : \theta = \theta_0$$
$$\text{versus}$$
$$H_a : \theta < \theta_0.$$

We will use the test statistic

$$Z = \frac{\overline{Y} - \theta_0}{\sqrt{\theta_0/n}}$$

and the rejection region RR $= \{z : z < -z_\alpha\}$, where z_α is the upper α quantile of the standard normal distribution.

(a) Show that, when H_0 is true, $Z \xrightarrow{d} \mathcal{N}(0,1)$, as $n \to \infty$. Conclude that RR is an approximate level α rejection region. *Hint:* This is a straightforward application of the CLT (but show the details).

(b) When H_a is true and $\theta = \theta_a < \theta_0$, we would like the probability of Type II Error to be β. Derive a formula for the necessary sample size n that maintains

$$\begin{aligned} P(\text{Reject } H_0 | H_0 \text{ is true}) &= \alpha \\ P(\text{Do not reject } H_0 | \theta = \theta_a) &= \beta. \end{aligned}$$

Your answer should depend on α, β, θ_0, and θ_a.

(c) Evaluate your formula in part (b) when $\alpha = 0.01$, $\beta = 0.1$, $\theta_0 = 30$, and $\theta_a = 28$.

Problem 5.

The Rockwell hardness index for steel is determined by pressing a diamond point into the steel and measuring the depth of penetration. For 50 specimens of an alloy of steel, the Rockwell hardness index averaged 62 with standard deviation 8. The manufacturer claims that this alloy has an average hardness index of at least 64. Is there sufficient evidence to refute the manufacturer's claim at the 1% significance level?

Problem 6.

The state of California is working very hard to ensure that all elementary age students whose native language is not English become proficient in English by the sixth grade. Their progress is monitored each year using the California English Language Development test. The results for two school districts in southern California for the 2003 school year are given in the accompanying table.[7] Do the data indicate a significant difference in the 2003 proportions of students who are fluent in English for the two districts? Use $\alpha = .01$.

District	Riverside	Palm Springs
Number of students tested	6124	5512
Percentage fluent	40	37