**HW 3 (Due Oct 3, 2017)** **Name:**

**Problem 1.**

Suppose that $Y_1, Y_2, ..., Y_n$ is an iid sample from

$$f_Y(y;\theta) = \begin{cases} \theta(1-y)^{\theta-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Derive the level $\alpha$ likelihood ratio test of $H_0 : \theta = 1$ versus $H_a : \theta \neq 1$.
(b) Derive the power function $K(\theta)$.
(c) Use R to plot $K(\theta)$ when $n = 10$ and $\alpha = 0.10$. Compute $K(0.8)$ and $K(1.5)$.

**Problem 2.**

Let $Y_1, Y_2, \ldots, Y_{20}$ be a random sample of size $n = 20$ from a normal distribution with unknown mean $\mu$ and known variance $\sigma^2 = 5$. We wish to test $H_0 : \mu = 7$ versus $H_a : \mu > 7$.

**a** Find the uniformly most powerful test with significance level .05.

**b** For the test in part (a), find the power at each of the following alternative values for $\mu :\ \mu_a = 7.5, 8.0, 8.5$, and $9.0$.

**c** Sketch a graph of the power function.

**Problem 3**.

Suppose that we have a random sample of four observations from the density function

$$f(y \mid \theta) = \begin{cases} \left(\dfrac{1}{2\theta^3}\right) y^2 e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

**a** Find the rejection region for the most powerful test of $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, assuming that $\theta_a > \theta_0$. [*Hint:* Make use of the $\chi^2$ distribution.]

**b** Is the test given in part (a) uniformly most powerful for the alternative $\theta > \theta_0$?

Obtain an expression for the power function $K(\theta)$ using the rejection region in part (b). Use R to graph the power function when $\alpha = 0.01, \theta_0 = 3$ and $n = 4$. Further, take $\alpha = 0.01$ and $\theta_0 = 0.03$ to find the smallest sample size $n$ that guarantees $K(4) \geq 0.90$.

**Problem 4.**

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Bernoulli-distributed population with parameter $p$. That is,

$$p(y_i \mid p) = p^{y_i}(1-p)^{1-y_i}, \qquad y_i = 0, 1.$$

**a** Suppose that we are interested in testing $H_0 : p = p_0$ versus $H_a : p = p_a$, where $p_0 < p_a$.

**i** Show that

$$\frac{L(p_0)}{L(p_a)} = \left[\frac{p_0(1-p_a)}{(1-p_0)p_a}\right]^{\sum y_i} \left(\frac{1-p_0}{1-p_a}\right)^n.$$

**ii** Argue that $L(p_0)/L(p_a) < k$ if and only if $\sum_{i=1}^n y_i > k^*$ for some constant $k^*$.

**iii** Give the rejection region for the most powerful test of $H_0$ versus $H_a$.

**b** Recall that $\sum_{i=1}^n Y_i$ has a binomial distribution with parameters $n$ and $p$. Indicate how to determine the values of any constants contained in the rejection region derived in part [a(iii)].

**c** Is the test derived in part (a) uniformly most powerful for testing $H_0 : p = p_0$ versus $H_a : p > p_0$? Why or why not?

Now suppose that $n = 60$ and $p_0 = 0.30$. In part (a), find the value of $k^*$ that makes $\alpha$ as close to 0.05 as possible. Further, use $n = 60$, $p_0 = 0.3$ and the $k^*$ you just obtained to find an expression for the power function $K(p)$. Use $R$ to graph the power function.

**Problem 5.**

Let $X_1, X_2, \ldots, X_m$ denote a random sample from the exponential density with mean $\theta_1$ and let $Y_1, Y_2, \ldots, Y_n$ denote an independent random sample from an exponential density with mean $\theta_2$.

**a** Find the likelihood ratio criterion for testing $H_0: \theta_1 = \theta_2$ versus $H_a: \theta_1 \neq \theta_2$.

**b** Show that the test in part (a) is equivalent to an exact $F$ test [*Hint:* Transform $\sum X_i$ and $\sum Y_j$ to $\chi^2$ random variables.]

**Problem 6**.

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a normal distribution with mean $\mu$ (unknown) and variance $\sigma^2$. For testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 > \sigma_0^2$, show that the likelihood ratio test is equivalent to the $\chi^2$ test

**Problem 7.**

Suppose that $Y_1, Y_2, ..., Y_5$ is an iid sample of size $n = 5$ from

$$f_Y(y;\theta) = \begin{cases} \theta e^{-\theta y}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the uniformly most powerful (UMP) level $\alpha = 0.10$ rejection region to test

$$\begin{gathered} H_0 : \theta = 1 \\ \text{versus} \\ H_a : \theta < 1. \end{gathered}$$

Your rejection region should not include any unknown constants. If it includes a quantile from a well known distribution, use properly defined notation to identify it.