**HW 7 (Due Dec 05, 2017)** **Name:**

**Problem 1.**

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from an exponentially distributed population with density $f(y \mid \theta) = \theta e^{-\theta y}, 0 < y$. (*Note*: the mean of this population is $\mu = 1/\theta$.) Use the conjugate gamma $(\alpha, \beta)$ prior for $\theta$ to do the following.

**a** Show that the joint density of $Y_1, Y_2, \ldots, Y_n, \theta$ is

$$f(y_1, y_2, \ldots, y_n, \theta) = \frac{\theta^{n+\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \exp\left[-\theta \bigg/ \left(\frac{\beta}{\beta \sum y_i + 1}\right)\right].$$

**b** Show that the marginal density of $Y_1, Y_2, \ldots, Y_n$ is

$$m(y_1, y_2, \ldots, y_n) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\beta^{\alpha}} \left(\frac{\beta}{\beta \sum y_i + 1}\right)^{\alpha+n}.$$

**c** Show that the posterior density for $\theta \mid (y_1, y_2, \ldots, y_n)$ is a gamma density with parameters $\alpha^\star = n + \alpha$ and $\beta^\star = \beta / (\beta \sum y_i + 1)$.

**d** Show that the Bayes estimator for $\mu = 1/\theta$ is

$$\hat{\mu}_B = \frac{\sum Y_i}{n + \alpha - 1} + \frac{1}{\beta(n + \alpha - 1)}.$$

[*Hint*: Recall Exercise 4.111(e).]

**e** Show that the Bayes estimator in part (d) can be written as a weighted average of $\overline{Y}$ and the prior mean for $1/\theta$. [*Hint*: Recall Exercise 4.111(e).]

**f** Show that the Bayes estimator in part (d) is a biased but consistent estimator for $\mu = 1/\theta$.

**g.** Assuming that a sample of size $n = 15$ produced a sample such that $\sum y_i = 30.27$ and the parameters of the gamma prior are $\alpha = 2.3$ and $\beta = 0.4$, use the R function *qgamma* to find a 95% credible intervals for $\theta$ and $1/\theta$. Further, conduce the Bayesian test for $H_0 : \theta > 2$ versus $H_1 : \theta \leq 2$. (part g requires what we will discuss on Tuesday's class. You do not need to finish it for HW 7, but should finish it before the final exam).

**Problem 2.**

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Poisson-distributed population with mean $\lambda$. In this case, $U = \sum Y_i$ is a sufficient statistic for $\lambda$, and $U$ has a Poisson distribution with mean $n\lambda$. Use the conjugate gamma $(\alpha, \beta)$ prior for $\lambda$ to do the following.

**a** Show that the joint likelihood of $U, \lambda$ is

$$L(u, \lambda) = \frac{n^u}{u!\beta^\alpha \Gamma(\alpha)} \lambda^{u+\alpha-1} \exp\left[-\lambda \Bigg/ \left(\frac{\beta}{n\beta + 1}\right)\right].$$

**b** Show that the marginal mass function of $U$ is

$$m(u) = \frac{n^u \Gamma(u + \alpha)}{u!\beta^\alpha \Gamma(\alpha)} \left(\frac{\beta}{n\beta + 1}\right)^{u+\alpha}.$$

**c** Show that the posterior density for $\lambda \mid u$ is a gamma density with parameters $\alpha^\star = u + \alpha$ and $\beta^\star = \beta/(n\beta + 1)$.

**d** Show that the Bayes estimator for $\lambda$ is

$$\hat{\lambda}_B = \frac{\left(\sum Y_i + \alpha\right)\beta}{n\beta + 1}.$$

**e** Show that the Bayes estimator in part (d) can be written as a weighted average of $\overline{Y}$ and the prior mean for $\lambda$.

**f** Show that the Bayes estimator in part (d) is a biased but consistent estimator for $\lambda$.

**g.** Assuming that a sample of size $n = 25$ produced a sample such that $\sum y_i = 174$ and the parameters of the gamma prior are $\alpha = 2$ and $\beta = 3$, use the R function *qgamma* to find a 95% credible intervals for $\lambda$. Further, conduce the Bayesian test for $H_0 : \lambda > 5$ versus $H_1 : \lambda \leq 5$. (again, part g requires what we will discuss on Tuesday's class. You do not need to finish it for HW 7, but should finish it before the final exam).