Stat 713 02/21/2019 UMVUE

Approach I. Crámer-Rao-Inequality

If $\hat{\theta}$ is an unbiased estimator of θ

$$Var(\hat{\theta}) \geq \frac{1}{n I_1(\theta)} \leftarrow \text{1st}$$

where $I_1(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \ln f_X(x|\theta)\right]$

If it is for $\tau(\theta)$

Then CRLB is $\dfrac{[\tau'(\theta)]^2}{n I_1(\theta)}$ $\leftarrow$ 2nd

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Attainment Corollary

$$= \int \frac{\partial}{\partial\theta} f_X(x|\theta)$$
$$= \frac{\partial}{\partial\theta} \int f_X(x|\theta)$$
$$= \frac{\partial}{\partial\theta} 1$$
$$= 0$$

Example 7.15. Suppose $X_1, X_2, ..., X_n$ are iid Poisson(θ), where $\theta > 0$. Find the CRLB on the variance of unbiased estimators of $\tau(\theta) = \theta$.
Solution. We know that the CRLB is

$$\frac{1}{I_n(\theta)} = \frac{1}{nI_1(\theta)},$$

where

$$I_1(\theta) = E_\theta\left\{\left[\frac{\partial}{\partial\theta}\ln f_X(X|\theta)\right]^2\right\} = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\ln f_X(X|\theta)\right].$$

For $x = 0, 1, 2, ...,$

$$\ln f_X(x|\theta) = \ln\left(\frac{\theta^x e^{-\theta}}{x!}\right) = x\ln\theta - \theta - \ln x!.$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial\theta}\ln f_X(x|\theta) &= \frac{x}{\theta} - 1 \\ \frac{\partial^2}{\partial\theta^2}\ln f_X(x|\theta) &= -\frac{x}{\theta^2}. \end{aligned}$$

The Fisher information based on one observation is

$$\begin{aligned} I_1(\theta) &= -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\ln f_X(X|\theta)\right] \\ &= -E_\theta\left(-\frac{X}{\theta^2}\right) = \frac{1}{\theta}. \end{aligned}$$

Therefore, the CRLB on the variance of all unbiased estimators of $\tau(\theta) = \theta$ is

$$\text{CRLB} = \frac{1}{nI_1(\theta)} = \frac{\theta}{n}. \quad = \text{Var}(\overline{X}_n)$$

Observation: Because $W(\mathbf{X}) = \overline{X}$ is an unbiased estimator of $\tau(\theta) = \theta$ in the Poisson(θ) model and because

$$\text{var}_\theta(\overline{X}) = \frac{\theta}{n},$$

we see that $\text{var}_\theta(\overline{X})$ does attain the CRLB. This means that $W(\mathbf{X}) = \overline{X}$ is the UMVUE for $\tau(\theta) = \theta$.

Example 7.16. Suppose $X_1, X_2, ..., X_n$ are iid gamma(α_0, β), where α_0 is known and $\beta > 0$. Find the CRLB on the variance of unbiased estimators of β.
Solution. We know that the CRLB is

$$\frac{1}{I_n(\beta)} = \frac{1}{nI_1(\beta)},$$

where

$$I_1(\beta) = E_\beta\left\{\left[\frac{\partial}{\partial\beta}\ln f_X(X|\beta)\right]^2\right\} = -E_\beta\left[\frac{\partial^2}{\partial\beta^2}\ln f_X(X|\beta)\right].$$

For $x > 0$,

$$\begin{aligned}\ln f_X(x|\beta) &= \ln\left[\frac{1}{\Gamma(\alpha_0)\beta^{\alpha_0}}x^{\alpha_0-1}e^{-x/\beta}\right]\\ &= \underline{-\ln\Gamma(\alpha_0) - \alpha_0\ln\beta + (\alpha_0-1)\ln x - \frac{x}{\beta}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial\beta}\ln f_X(x|\beta) &= -\frac{\alpha_0}{\beta}+\frac{x}{\beta^2}\\ \frac{\partial^2}{\partial\beta^2}\ln f_X(x|\beta) &= \frac{\alpha_0}{\beta^2}-\frac{2x}{\beta^3}.\end{aligned}$$

The Fisher information based on one observation is

$$\begin{aligned}\underline{I_1(\beta)} &= -E_\beta\left[\frac{\partial^2}{\partial\beta^2}\ln f_X(X|\beta)\right]\\ &= \underline{-E_\beta\left(\frac{\alpha_0}{\beta^2}-\frac{2X}{\beta^3}\right) = \frac{\alpha_0}{\beta^2}}.\end{aligned}$$

Therefore, the CRLB on the variance of all unbiased estimators of β is

$$\boxed{\text{CRLB} = \frac{1}{nI_1(\beta)} = \frac{\beta^2}{n\alpha_0}.} \quad = Var(\overline{X}_n/\alpha_0)$$

Observation: Consider the estimator

$$W(\mathbf{X}) = \frac{\overline{X}}{\alpha_0}.$$

Note that

$$E_\beta[W(\mathbf{X})] = E_\beta\left(\frac{\overline{X}}{\alpha_0}\right) = \frac{\alpha_0\beta}{\alpha_0} = \beta$$

and

$$\text{var}_\beta[W(\mathbf{X})] = \text{var}_\beta\left(\frac{\overline{X}}{\alpha_0}\right) = \frac{\alpha_0\beta^2}{n\alpha_0^2} = \frac{\beta^2}{n\alpha_0}.$$

We see that $W(\mathbf{X}) = \overline{X}/\alpha_0$ is an unbiased estimator for β and $\text{var}_\beta(\overline{X}/\alpha_0)$ attains the CRLB. This means that $W(\mathbf{X}) = \overline{X}/\alpha_0$ is the UMVUE for β.

Discussion: Instead of estimating β in Example 7.16, suppose that we were interested in estimating $\tau(\beta) = 1/\beta$ instead.

known. $\frac{\overline{X}_n}{\alpha_0}$ is the UMVUE for β

1. Show that
$$W(\mathbf{X}) = \frac{n\alpha_0 - 1}{n\overline{X}}$$
is an unbiased estimator of $\tau(\beta) = 1/\beta$.

2. Derive the CRLB for the variance of unbiased estimators of $\tau(\beta) = 1/\beta$. ???

3. Calculate $\text{var}_\beta[W(\mathbf{X})]$ and show that it is strictly larger than the CRLB (i.e., the variance does not attain the CRLB).

Q: Does this necessarily imply that $W(\mathbf{X})$ cannot be the UMVUE of $\tau(\beta) = 1/\beta$?

Remark: In general, the CRLB offers a **lower bound** on the variance of any unbiased estimator of $\tau(\theta)$. However, this lower bound may be unattainable. That is, the CRLB may be strictly smaller than the variance of any unbiased estimator. If this is the case, then our "CRLB approach" to finding an UMVUE will not be helpful.

Corollary 7.3.15 (Attainment). Suppose $X_1, X_2, ..., X_n$ is an iid sample from $f_X(x|\theta)$, where $\theta \in \Theta$, a family that satisfies the regularity conditions stated for the Cramér-Rao Inequality. If $W(\mathbf{X})$ is an unbiased estimator of $\tau(\theta)$, then $\text{var}_\theta[W(\mathbf{X})]$ attains the CRLB if and only if the score function
$$S(\theta|\mathbf{x}) = a(\theta)[W(\mathbf{x}) - \tau(\theta)]$$
is a linear function of $W(\mathbf{x})$.

Recall: The **score function** is given by
$$\begin{aligned} S(\theta|\mathbf{x}) &= \frac{\partial}{\partial\theta} \ln L(\theta|\mathbf{x}) \\ &= \frac{\partial}{\partial\theta} \ln f_{\mathbf{X}}(\mathbf{x}|\theta). \end{aligned}$$

Example 7.16 (continued). Suppose $X_1, X_2, ..., X_n$ are iid gamma(α_0, β), where α_0 is known and $\beta > 0$. The likelihood function is
$$\begin{aligned} L(\beta|\mathbf{x}) &= \prod_{i=1}^n \frac{1}{\Gamma(\alpha_0)\beta^{\alpha_0}} x_i^{\alpha_0 - 1} e^{-x_i/\beta} \\ &= \left[\frac{1}{\Gamma(\alpha_0)\beta^{\alpha_0}}\right]^n \left(\prod_{i=1}^n x_i\right)^{\alpha_0 - 1} e^{-\sum_{i=1}^n x_i/\beta}. \end{aligned}$$

The log-likelihood function is
$$\ln L(\beta|\mathbf{x}) = -n\ln\Gamma(\alpha_0) - n\alpha_0 \ln\beta + (\alpha_0 - 1)\sum_{i=1}^n \ln x_i - \frac{\sum_{i=1}^n x_i}{\beta}.$$

The score function is

$$\begin{aligned} S(\beta|\mathbf{x}) = \frac{\partial}{\partial\beta}\ln L(\beta|\mathbf{x}) &= -\frac{n\alpha_0}{\beta} + \frac{\sum_{i=1}^n x_i}{\beta^2} \\ &= \frac{n\alpha_0}{\beta^2}\left(\frac{\sum_{i=1}^n x_i}{n\alpha_0} - \beta\right) \\ &= a(\beta)[W(\mathbf{x}) - \tau(\beta)], \end{aligned}$$

where

$$W(\mathbf{x}) = \frac{\sum_{i=1}^n x_i}{n\alpha_0} = \frac{\overline{x}}{\alpha_0}.$$

We have written the score function $S(\beta|\mathbf{x})$ as a linear function of $W(\mathbf{x}) = \overline{x}/\alpha_0$. Because $W(\mathbf{X}) = \overline{X}/\alpha_0$ is an unbiased estimator of $\tau(\beta) = \beta$ (shown previously), the variance $\text{var}_\beta[W(\mathbf{X})]$ attains the CRLB for the variance of unbiased estimators of $\tau(\beta) = \beta$.

Remark: The attainment result is interesting, but I have found that its usefulness may be limited if you want to find the UMVUE. Even if we can write

$$S(\theta|\mathbf{x}) = a(\theta)[W(\mathbf{x}) - \tau(\theta)]$$

where $E_\theta[W(\mathbf{X})] = \tau(\theta)$, the RHS might involve a function $\tau(\theta)$ for which there is no desire to estimate. To illustrate this, suppose $X_1, X_2, ..., X_n$ are iid beta$(\theta, 1)$, where $\theta > 0$. The score function is

$$\begin{aligned} S(\theta|\mathbf{x}) &= \frac{n}{\theta} + \sum_{i=1}^n \ln x_i \\ &= n\left[\frac{\sum_{i=1}^n \ln x_i}{n} - \left(-\frac{1}{\theta}\right)\right] \\ &= a(\theta)[W(\mathbf{x}) - \tau(\theta)]. \end{aligned}$$

It turns out that

$$E_\theta[W(\mathbf{X})] = E_\theta\left(\frac{1}{n}\sum_{i=1}^n \ln X_i\right) = -\frac{1}{\theta}.$$

We have shown that $\text{var}_\theta[W(\mathbf{X})]$ attains the CRLB on the variance of unbiased estimators of $\tau(\theta) = -1/\theta$, a parameter we likely have no desire to estimate.

Unresolved issues:

1. What if $f_X(x|\theta)$ does not satisfy the regularity conditions needed for the Cramér-Rao Inequality to apply? For example, $X \sim \mathcal{U}(0, \theta)$.

2. What if the CRLB is unattainable? Can we still find the UMVUE?

7.3.3 Sufficiency and completeness

Remark: We now move to our "second approach" on how to find UMVUEs. This approach involves sufficiency and completeness—two topics we discussed in the last chapter. We can also address the unresolved issues on the previous page.

Theorem 7.3.17 (**Rao-Blackwell**). Let $W = W(\mathbf{X})$ be an unbiased estimator of $\tau(\theta)$. Let $T = T(\mathbf{X})$ be a sufficient statistic for θ. Define

Is $\phi(T)$ a statistic??? Yes

$$\phi(T) = E(W|T).$$

$\phi(T) = E[W(\underset{\sim}{X}) \,|\, T(\underset{\sim}{X})]$

Then

$= \int W(\underset{\sim}{x}) f_{\underset{\sim}{X}|T}(\underset{\sim}{x}|T)\, d\underset{\sim}{x}$

1. $E_\theta[\phi(T)] = \tau(\theta)$ for all $\theta \in \Theta$
2. $\text{var}_\theta[\phi(T)] \le \text{var}_\theta(W)$ for all $\theta \in \Theta$.

If $\phi(T)$ unbiased for $\tau(\theta)$?

That is, $\phi(T) = E(W|T)$ is a uniformly better unbiased estimator than W.
Proof. This result follows from the iterated rules for means and variances. First,

$$E_\theta[\phi(T)] = E_\theta[E(W|T)] = E_\theta(W) = \tau(\theta).$$

$E[\phi(T)] = E[E(W|T)]$
$= E(W)$
$= \tau(\theta)$

Second,

$$\begin{aligned}
\text{var}_\theta(W) &= E_\theta[\text{var}(W|T)] + \text{var}_\theta[E(W|T)] \\
&= E_\theta[\text{var}(W|T)] + \text{var}_\theta[\phi(T)] \\
&\ge \text{var}_\theta[\phi(T)],
\end{aligned}$$

because $\text{var}(W|T) \ge 0$ (a.s.) and hence $E_\theta[\text{var}(W|T)] \ge 0$. $\square$

Implication: We can always "improve" the unbiased estimator W by conditioning on a sufficient statistic.

Remark: To use the Rao-Blackwell Theorem, some students think they have to

1. Find an unbiased estimator W.
2. Find a sufficient statistic T.
3. Derive the conditional distribution $f_{W|T}(w|t)$.
4. Find the mean $E(W|T)$ of this conditional distribution.

This is not the case at all! Because $\phi(T) = E(W|T)$ is a function of the sufficient statistic T, the Rao-Blackwell result simply convinces us that in our search for the UMVUE, we can restrict attention to those estimators that are functions of a sufficient statistic.