Preliminary 1. Convergence in real number.

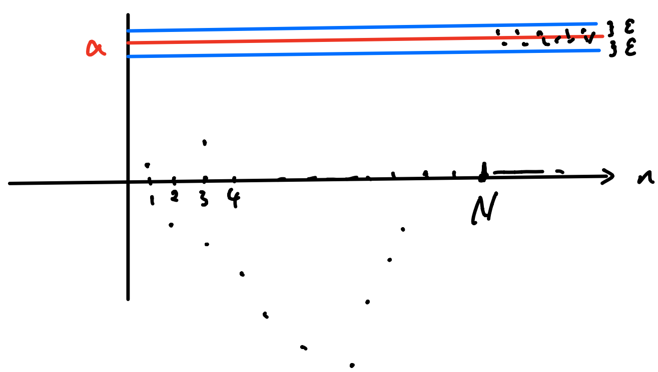
$\{a_n\}_{n=1}^{+\infty}$, $a \in R$

$a_n \to a$ as $n \to +\infty$ or $\lim_{n\to\infty} a_n = a$

if $\forall \varepsilon > 0$, $\exists N$ such that $\forall n > N$

$|a_n - a| < \varepsilon$

$\varepsilon > 0$



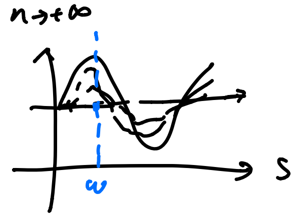
Prelim 2. Pointwise convergence of a sequence of real functions.

f_n : Domain $S \to R$

f : Domain $S \to R$

if $\forall \omega \in S$ $\quad f_n(\omega) \to f(\omega)$ as $n \to +\infty$

$\{a_n\}_{n=1}^{+\infty} \to a$



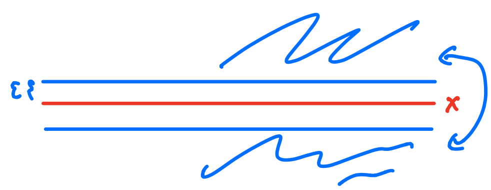
$f_n \to f$ pointwisely as $n \to +\infty$

- $X_n \to X$?

1. Convergence in Probability $X_n \xrightarrow{P} X$

$\forall \varepsilon > 0$, $$P(\underbrace{|X_n - X|}_{Y_n =: X_n - X} > \varepsilon) \to 0$$

$$X_n \xrightarrow{P} X \iff Y_n \xrightarrow{P} 0$$



4 Approaches

① Defi. $\underbrace{P(|X_n - X| > \varepsilon)}_{\to 0 \text{ as } n \to +\infty}$ $\forall \varepsilon > 0$

② Markov's Ineq

$$P(|X_n - X| > \varepsilon) \le \frac{E|X_n - X|^k}{\varepsilon^k}$$

show $E|X_n - X|^k \to 0$ as $n \to +\infty$

(often take $k=2$)

③ WLLN $Y_1 \cdots Y_n \overset{iid}{\sim} Y$ $E|Y| < +\infty$

or $Var(Y) < +\infty$

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{P} E[Y]$$

④ $X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{P} c$

Properties: $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} Y$, $C_n \to C$

also hold for $\xrightarrow{a.s.}$

- $c X_n \xrightarrow{P} cX$, $C_n X_n \xrightarrow{P} cX$
- $X_n \pm Y_n \xrightarrow{P} X \pm Y$
- $X_n \cdot Y_n \xrightarrow{P} XY$
- $X_n / Y_n \xrightarrow{P} X/Y$ provided $P(Y=0)=0$
- If h is continuous $h(X_n) \xrightarrow{P} h(X)$. (Continuous Mapping)

$R \to R$

↑

range of X_n, X

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2. Almost sure convergence. (converge almost surely)

$$X_n \xrightarrow{a.s.} X \quad \text{as } n \to +\infty$$

if $\forall \varepsilon > 0$, $$P\left(\lim_{n\to\infty} |X_n - X| < \varepsilon \right) = 1$$

or simply: $$P\left(\lim_{n\to\infty} X_n = X \right) = 1$$

$$\left\{ \text{Continuous Mapping} \quad \text{If } h \text{ is continuous } X_n \xrightarrow{a.s} X \Rightarrow h(X_n) \xrightarrow{a.s.} h(X) \right\}$$

$(S, \mathcal{B}, P)$ $\qquad$ $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$

induced Prob.

$$X: S \to \mathbb{R}$$

$$P_X(X \in B) = P(\{w \in S, \ X(w) \in B\})$$

Eg. $\quad X: \begin{array}{l} \text{Head} \to 1 \\ \text{Tail} \to 0 \end{array}$ $\quad$ fair coin

$$P_X(X=1) = P(\{\text{Head}\}) = \frac{1}{2}$$

Now, we have $\quad X_n: S \to \mathbb{R} \quad n=1, \dots, n$

$$X: S \to \mathbb{R}$$

$\forall w \in S \quad X_n(w) \to X(w)$

or $\quad X_n(w) \not\to X(w)$

pick all w such that $\quad X_n(w) \to X(w)$

$$\Omega = \{w \in S: \ X_n(w) \to X(w)\}$$

If $P(\Omega) = 1$, $\quad X_n \xrightarrow{a.s.} X$

eg. $\quad X_n: \begin{array}{l} \text{head} \to 1 + \frac{1}{n} \\ \text{Taid} \to 0 - \frac{1}{n} \end{array}$ $\qquad X: \begin{array}{l} \text{head} \to 1 \\ \text{Tail} \to 0 \end{array}$

for "head" $X_n(\text{head}) = 1+\frac{1}{n} \to 1 = X(\text{head})$

for "Tail". $X_n(\text{tail}) = 0-\frac{1}{n} \to 0 = X(\text{tail})$

$$\Omega = \{w \in S: X_n(w) \to X(w)\}$$

$$= \{\text{head}, \text{tail}\} = S$$

$$P(\Omega) = P(S) = 1.$$

$$X_n \xrightarrow{a.s.} X$$

Study Textbook Eample 5.5.7, 5.5.8

5.5.7 ↓ explains difference between almost surley convergence and pointwisly convergence of $f_n \to f$

5.5.8 → differencn between $X_n \xrightarrow{a.s} X$ and $X_n \xrightarrow{P} X$.

Approach: SLLN

$Y_1 \cdots Y_n \overset{iid}{\sim} Y$ $\quad Var(Y) < +\infty$

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{a.s.} E[Y]$$

$$X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X$$

- Convergence in distribution: (or in law or weakly converge)

$$X_n \xrightarrow{d} X \qquad (X_n \xrightarrow{L} X)$$

if $\lim_{n\to\infty} \underbrace{F_{X_n}(x)}_{\text{cdf of } X_n} = \underbrace{F_X(x)}_{\text{cdf of } X}$ at all points x where F_X is continuous.

(Tricky) Eg

$X_n = Z \sim N(0,1)$ $\quad F_{X_n}$

$X = -Z \sim N(0,1)$ $\quad \| \; F_X$

$X_n \xrightarrow{d} X \qquad X_n - X = 2Z \not\to 0$

$$\begin{cases} X_n \xrightarrow{a.s.} X & \Rightarrow \quad X_n - X \xrightarrow{a.s.} 0 \\ X_n \xrightarrow{P} X & \Rightarrow \quad X_n - X \xrightarrow{P} 0 \\ X_n \xrightarrow{d} X & \not\Rightarrow \quad X_n - X \xrightarrow{d} 0 \end{cases}$$

- $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$

- If $X_n \xrightarrow{d} c \Leftrightarrow X_n \xrightarrow{P} c$

- CLT $\quad Y_1 \cdots Y_n \overset{iid}{\sim} Y. \qquad Var(Y) = \sigma^2 < +\infty$
  $E(Y) = \mu < +\infty$

$$\sqrt{n}(\bar{Y}_n - \mu) \xrightarrow{d} N(0, \sigma^2) \quad \text{as } n\to+\infty$$

$$\frac{\sqrt{n}(\bar{Y}_n-\mu)}{\sigma} \xrightarrow{d} N(0,1)$$

$$\frac{\sqrt{n}(\bar{Y}_n-\mu)}{\sigma} \sim \underset{\substack{\uparrow \\ \text{asymptotically}}}{A}N(0,1)$$

[MGF approach to prove CLT, or Characteristic Func.]

$X_n \xrightarrow{d} X$, $Y_n \xrightarrow{d} Y$, $C_n \to C$

$Z \sim N(0,1)$

- $cX_n \xrightarrow{d} cX$, $\quad C_n X_n \xrightarrow{d} cX$
- $\cancel{X_n \pm Y_n \xrightarrow{d} X \pm Y}$ ← conterexample

$X_n = Z \xrightarrow{d} Z$

$Y_n = -Z \xrightarrow{d} Z$

$X_n + Y_n = 0 \xrightarrow{d} 2Z$ ← not right

- $\cancel{X_n \cdot Y_n \xrightarrow{d} XY}$
- $\cancel{X_n / Y_n \xrightarrow{d} X/Y}$ ~~provided~~ $\cancel{P(Y=0)=0}$
- If h is continuous $R \to R$ (range of X_n, X) $\quad h(X_n) \xrightarrow{d} h(X)$. (Continuous Mapping)

eg. If $X_n \xrightarrow{d} N(0,1)$

$X_n^2 \xrightarrow{d} \chi_1^2$

$h(x) = x^2$ is continuous

$\sin X_n \xrightarrow{d} \sin(N(0,1))$

Slutsky's theorem: if $X_n \xrightarrow{d} X \quad Y_n \xrightarrow{P} a$ ← constant

Then
- $X_n Y_n \xrightarrow{d} aX$
- $X_n \pm Y_n \xrightarrow{d} X \pm a$

Eg. $X_1, \cdots, X_n \overset{iid}{\sim} X \qquad E[X] = \mu, \quad Var[X] = \sigma^2$

$$\begin{cases} \dfrac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1) & \left[\bar{X}_n \pm Z_{\alpha/2} \dfrac{\sigma}{\sqrt{n}}\right] \\ S_n \xrightarrow{P} \sigma & \left[\bar{X}_n \pm Z_{\alpha/2} \dfrac{S_n}{\sqrt{n}}\right] \end{cases}$$

S_n: sample standard deviation

$$\frac{\sqrt{n}(\bar{X}_n-\mu)}{S_n} = \underbrace{\frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma}}_{\substack{\downarrow d \\ N(0,1)}} \times \underbrace{\frac{\sigma}{S_n}}_{\substack{\downarrow p \\ 1}} \xrightarrow{d} N(0,1)\times 1 = N(0,1)$$

Eg. $Y_1, \cdots, Y_n \overset{iid}{\sim}$ Bernoulli (p)

$$\begin{cases} \dfrac{\sqrt{n}(\bar{Y}_n - p)}{\sqrt{p(1-p)}} \xrightarrow{d} N(0,1) \\ \bar{Y}_n \xrightarrow{p} p \Rightarrow \sqrt{\bar{Y}_n(1-\bar{Y}_n)} \xrightarrow{p} \sqrt{p(1-p)} \end{cases}$$

$\uparrow$ continuous map $\quad h(x) = \sqrt{x(1-x)}$

$$\Rightarrow \frac{\sqrt{n}(\bar{Y}_n - p)}{\sqrt{\bar{Y}_n(1-\bar{Y}_n)}} = \underbrace{\frac{\sqrt{n}(\bar{Y}_n-p)}{\sqrt{p\times(1-p)}}}_{\substack{\downarrow d \\ N(0,1)}} \times \underbrace{\frac{\sqrt{p\times(1-p)}}{\sqrt{\bar{Y}_n(1-\bar{Y}_n)}}}_{\substack{\downarrow p \\ 1}}$$

$$\xrightarrow{d} N(0,1)$$