Z-Interval for population mean μ

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma_0^2)$ with σ_0^2 known

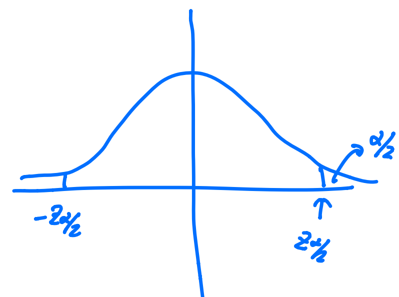
$100 \times (1-\alpha)\%$ two-sided confidence interval for μ

$$\left[\overline{X}_n - z_{\alpha/2} \frac{\sigma_0}{\sqrt{n}}, \overline{X}_n + z_{\alpha/2} \frac{\sigma_0}{\sqrt{n}} \right]$$

$$P\left(\overline{X}_n - z_{\alpha/2} \frac{\sigma_0}{\sqrt{n}} \le \mu \le \overline{X}_n + z_{\alpha/2} \frac{\sigma_0}{\sqrt{n}} \right) = 1-\alpha$$

$\overline{X}_n \sim N\left(\mu, \frac{\sigma_0^2}{n}\right)$ sampling distr. of $\overline{X}_n$

$$\frac{\overline{X}_n - \mu}{\sigma_0/\sqrt{n}} \sim N(0,1)$$



$$P\left(-z_{\alpha/2} \le \frac{\overline{X}_n - \mu}{\sigma_0/\sqrt{n}} \le z_{\alpha/2} \right) = 1-\alpha$$

Suppose the population distribution is not normal

$X_1, \ldots, X_n \overset{iid}{\sim} E[X] = \mu$, $Var[X] = \sigma_0^2$ known

CLT:
$$\frac{(\overline{X}_n - \mu)}{\sigma_0/\sqrt{n}} \xrightarrow{d} N(0,1)$$

$$P\left(-z_{\alpha/2} \le \frac{\overline{X}_n - \mu}{\sigma_0/\sqrt{n}} \le z_{\alpha/2} \right) \to 1-\alpha \quad \text{as } n \to +\infty$$

$\left[\overline{X}_n \pm z_{\alpha/2} \frac{\sigma_0}{\sqrt{n}} \right]$ is an approximate $100 \times (1-\alpha)\%$ CI for μ

Delta Method.

$X_1 \cdots X_n \overset{iid}{\sim} E[X]=\mu, \quad Var[X]=\sigma^2 < +\infty$

μ^2

$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1)$

$\bar{X}_n \xrightarrow{P} \mu \quad$ WLLN

$\bar{X}_n^2 \xrightarrow{P} \mu^2$

$\frac{\sqrt{n}(\bar{X}_n^2 - \mu^2)}{??} \xrightarrow{d} N(0,1)$

$\frac{\sqrt{n}(\bar{X}_n^2 - \mu \times \bar{X}_n)}{\sigma} \xrightarrow{d} N(0, \mu^2)$

$$\boxed{\begin{gathered}\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2) \\ \mu^2 \quad g(x)=x^2 \quad g'(\mu)=2\mu \neq 0 \\ \sqrt{n}(\bar{X}_n^2 - \mu^2) = \sqrt{n}(g(\bar{X}_n) - g(\mu)) \\ \xrightarrow{d} N(0, (2\mu)^2 \sigma^2)\end{gathered}}$$

$\sqrt{n}(g(\bar{X}_n) - g(\mu)) \xrightarrow{d} ???$ $\leftarrow$ why Delta Method!

Theorem 5.5.24 Suppose we have a sequence of random variables $\{X_n\}_{n=1}^{+\infty}$

$$\sqrt{n}(X_n - \theta) \xrightarrow{d} N(0, \sigma^2)$$

Suppose $g: R \to R$ is a differentiable at θ and $g'(\theta) \neq 0$

Then $$\sqrt{n}(g(X_n) - g(\theta)) \xrightarrow{d} N(0, [g'(\theta)]^2 \sigma^2)$$

Proof: $$g(X_n) = g(\theta) + g'(\theta)(X_n - \theta) + \frac{g''(\xi)}{2}(X_n - \theta)^2$$

$$\sqrt{n}\left(g(X_n)-g(\theta)\right) = g'(\theta)\cdot \underbrace{\sqrt{n}(X_n-\theta)}_{\to N(0,\sigma^2)} \Big\} \xrightarrow{d} N(0, [g'(\theta)]^2\sigma^2)$$

$$+ \underbrace{\frac{g''(\varepsilon)}{2} \underbrace{(X_n-\theta)}_{\downarrow p \atop 0} \underbrace{\sqrt{n}(X_n-\theta)}_{\downarrow d \atop N(0,\sigma^2)}} \xrightarrow{p} 0$$

① $\sqrt{n}(X_n-\theta) \xrightarrow{d} N(0,\sigma^2)$

$\Downarrow$

② $X_n \xrightarrow{p} \theta \Leftarrow P(|X_n-\theta|>\varepsilon) \to 0$

$\Uparrow$

$P(|\underbrace{\sqrt{n}(X_n-\theta)}|>\sqrt{n}\,\varepsilon) \to 0$

$\Uparrow$ $P(|N(0,\sigma^2)|>\sqrt{n}\,\varepsilon) \to 0$

③ $\frac{g''(\varepsilon)}{2} \xrightarrow{p} \frac{g''(\theta)}{2}$ ε between X_n and θ

Second-Order Delta Method

$$\sqrt{n}(X_n-\theta) \xrightarrow{d} N(0,\sigma^2) \quad \text{as } n\to+\infty$$

$g: R\to R$. differentiable at θ

$g'(\theta)=0$

$g''(\theta)\neq 0$

Then $$n\left[g(X_n)-g(\theta)\right] \xrightarrow[\text{as } n\to+\infty]{d} \frac{\sigma^2}{2} g''(\theta)\,\chi_1^2$$

Eg: for $\mu=0$ $\quad n(\bar{X}_n^2 - 0^2) \xrightarrow{d} \frac{\sigma^2}{2}\, 2 \times \chi_1^2 = \sigma^2\chi_1^2$

$g(x)=x^2 \quad g'(x)=2x \quad g''(x)=2$

$$\sqrt{n}(\bar{X}_n^2-\mu^2)=\sqrt{n}(g(\bar{X}_n)-g(\mu)) \xrightarrow{d} N(0,(2\mu)^2\sigma^2)$$

$$\frac{\sqrt{n}\,(\bar{X}_n^2-\mu^2)}{2\mu\sigma} \xrightarrow{d} N(0,1)$$

$$-z_{\alpha/2} \le \frac{\sqrt{n}\,(\bar{X}_n^2-\mu^2)}{2\mu\sigma} \le z_{\alpha/2}$$

$$\left[\bar{X}_n^2 \pm z_{\alpha/2}\cdot\frac{2\mu\sigma}{\sqrt{n}}\right] \quad \mu^2 \quad \times$$

$$\left[\bar{X}_n^2 \pm z_{\alpha/2}\times\frac{2\bar{X}_n S_n}{\sqrt{n}}\right] \quad \checkmark$$

Variance Stablization.

Examp 5.23 $\quad X_1 \cdots X_n \overset{iid}{\sim} \text{Poisson}(\theta)$

$$\sqrt{n}(\bar{X}_n-\theta) \xrightarrow{d} N(0,\ \theta\ )$$

Find a g such that

$$\sqrt{n}(g(\bar{X}_n)-g(\theta)) \xrightarrow{d} N(0,\ *\ )$$

fn of θ

$$\sqrt{n}\,(g(\bar{X}_n)-g(\theta)) \xrightarrow{d} N(0,\underbrace{[g'(\theta)]^2\theta}_{\text{free of }\theta})$$

e.g. Taking $g'(\theta)=\frac{1}{\sqrt{\theta}}$ $\Rightarrow [g'(\theta)]^2\theta = 1$

$$g(\theta) = 2\sqrt{\theta}$$

$$\sqrt{n}\left(2\sqrt{\bar{X}_n} - 2\sqrt{\theta}\right) \xrightarrow{d} N(0,1)$$

Page 177 Exercise.

Multivariate Case:

$$\underset{\sim}{X}_1 = \begin{pmatrix} X_{11} \\ \vdots \\ X_{1K} \end{pmatrix} \quad \underset{\sim}{X}_2 = \begin{pmatrix} X_{21} \\ \vdots \\ X_{2K} \end{pmatrix} \cdots \underset{\sim}{X}_n = \begin{pmatrix} X_{n1} \\ \vdots \\ X_{nK} \end{pmatrix}$$

$$\underset{\sim}{iid} \quad \underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_K \end{pmatrix} \qquad E(\underset{\sim}{X}) = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_K \end{pmatrix} = \underset{\sim}{\mu}$$

$$Cov(\underset{\sim}{X}) = \underset{\sim}{\Sigma}_{K\times K}$$

Multivariate CLT:

$$\underset{\sim}{\bar{X}_n} = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} X_{i1} \\ \vdots \\ \frac{1}{n}\sum_{i=1}^{n} X_{iK} \end{pmatrix} = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{X_i}$$

$$\sqrt{n}\left(\underset{\sim}{\bar{X}_n} - \underset{\sim}{\mu}\right) \xrightarrow{d} MVN\left(\underset{\sim}{0}, \underset{\sim}{\Sigma}_{K\times K}\right)$$

Delta Method

$g: R^k \to R$

$g(x_1, \cdots x_k) = \sum_{i=1}^{k} x_i$

$R^k \to R$

differentiable at μ (partial derivatives are not zero)

$$\sqrt{n}\left(\underbrace{g(\bar{X}_n)}_{\text{in } R} - \underbrace{g(\mu)}_{\text{in } R} \right) \xrightarrow{d} N\left(0, \ * \ \right)$$

$$* = \left(\frac{\partial g(x)}{\partial x_1}, \frac{\partial g(x)}{\partial x_2}, \cdots, \frac{\partial g(x)}{\partial x_k} \right) \Bigg|_{x=\mu}$$

$$\times \quad \Sigma_{k \times k}$$

$$\times \left(\begin{matrix} \frac{\partial g(x)}{\partial x_1} \\ \vdots \\ \frac{\partial g(x)}{\partial x_k} \end{matrix} \right) \Bigg|_{x=\mu}$$