Exponential family (Theorem 6.2.10)

If $X_1, \cdots, X_n \overset{iid}{\sim} f(x|\underset{\sim}{\theta}) = h(x)\, c(\underset{\sim}{\theta}) \exp\left(\sum_{j=1}^{k} w_j(\underset{\sim}{\theta})\, t_j(x)\right)$

$$\underset{\sim}{\theta} = (\theta_1, \cdots, \theta_d)^T \quad , d \le k$$

Then $T(\underset{\sim}{X}) = \begin{pmatrix} \sum_{i=1}^{n} t_1(X_i) \\ \vdots \\ \sum_{i=1}^{n} t_k(X_i) \end{pmatrix}$ is a sufficient statistic for $\underset{\sim}{\theta}$

Ex Gamma (α, β)

$N(\mu, \sigma^2) \leftarrow \quad T = T(\underset{\sim}{X}) = \begin{pmatrix} \sum_{i=1}^{n} X_i \\ \sum_{i=1}^{n} X_i^2 \end{pmatrix}$

if σ^2 is known, $\quad N(\mu, \sigma^2 = \sigma_0^2)$, $\theta = \mu \leftarrow T(\underset{\sim}{X}) = \sum_{i=1}^{n} X_i$

if μ is known $\quad N(\mu = \mu_0, \sigma^2)$. $\theta = \sigma^2 \leftarrow T(\underset{\sim}{X}) = \sum_{i=1}^{n} (X_i - \mu_0)^2$

Minimal Sufficient Statistics

Eample. $\quad N(\mu, \sigma_0^2) \quad \theta = \mu, \ \sigma_0^2$ known

① $\sum_{i=1}^{n} X_i \leftarrow \underline{\sum X_i + 1 \quad \exp(\sum X_i) \quad r(\sum_{i=1}^{n} X_i) \text{ where } r \text{ is one-to-one.}}$

② $(X_{(1)}, \cdots, X_{(n)})$

Because $\sum_{i=1}^{n} X_i = r\,(X_{(1)}, \cdots, X_{(n)}) = \sum_{i=1}^{n} X_{(i)}$

but $(X_{(1)}, \cdots, X_{(n)})$ cannot be obtained from $\sum X_i$

intuitively, $\sum X_i$ achieves more data reduction.

Definition. A sufficient statistic $T = T(\underset{\sim}{X})$ is called a minimum sufficient statistic if.

for all other sufficient statistic $T^*(\underset{\sim}{X})$

$T(\underset{\sim}{x})$ is a function of $T^*(\underset{\sim}{x})$

Theorem 6.2.13 $X_1, \cdots, X_n \overset{iid}{\sim} f(x|\theta)$, $f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$

Suppose there is a function $T(\underset{\sim}{x})$ such that for all $\underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{X}$

$$\frac{f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta)}{f_{\underset{\sim}{X}}(\underset{\sim}{y}|\theta)} \text{ is free of } \theta \iff T(\underset{\sim}{x}) = T(\underset{\sim}{y})$$

Then $T(\underset{\sim}{X})$ is a minimal sufficient statistic.

Eg. $N(\mu, \sigma_0^2)$ $\theta = \mu$. $f(x|\theta) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{(x-\mu)^2}{2\sigma_0^2}\right\}$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma_0^2}\right)$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n \exp\left\{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma_0^2}\right\}$$

$\underset{\sim}{x} = (x_1, \cdots, x_n)$ $\underset{\sim}{y} = (y_1, \cdots, y_n)$

$$\frac{f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta)}{f_{\underset{\sim}{X}}(\underset{\sim}{y}|\theta)} = \frac{\left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n \exp\left\{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma_0^2}\right\}}{\left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n \exp\left\{-\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{2\sigma_0^2}\right\}}$$

$$= \exp\left\{ \frac{\sum_{i=1}^{n}(y_i-\mu)^2 - \sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma_0^2} \right\}$$

$$= \exp\left\{ \frac{\sum_{i=1}^{n}(y_i^2-2\mu y_i+\mu^2) - \sum_{i=1}^{n}(x_i^2-2\mu x_i+\mu^2)}{2\sigma_0^2} \right\}$$

$$= \exp\left\{ \frac{\sum_{i=1}^{n}(y_i^2-x_i^2) - 2\mu\left(\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i\right)}{2\sigma_0^2} \right\}$$

is free of μ if and only if $\underbrace{\sum_{i=1}^{n} y_i}_{T(\underset{\sim}{y})} = \underbrace{\sum_{i=1}^{n} x_i}_{T(\underset{\sim}{x})}$

$T(\underset{\sim}{X}) = \sum_{i=1}^{n} X_i$ is minimal sufficient for $\theta = \mu$

Ex $\quad X_1, \cdots, X_n \overset{iid}{\sim} U(\theta, \theta+1)$

$$f_X(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} I(\theta < X_i < \theta+1)$$

$$= I(\theta < X_{(1)})\, I(X_{(n)} < \theta+1)$$

$$\frac{f_X(\underset{\sim}{x}|\theta)}{f_X(\underset{\sim}{y}|\theta)} = \frac{I(\theta < x_{(1)})\, I(X_{(n)} < \theta+1)}{I(\theta < y_{(1)})\, I(y_{(n)} < \theta+1)} \text{ is free of } \theta$$

if and only if $\quad X_{(1)} = Y_{(1)}$ and $X_{(n)} = Y_{(n)}$

$$T(\underset{\sim}{X}) = \begin{pmatrix} X_{(1)} \\ X_{(n)} \end{pmatrix}$$ is a minimal suff stat. for θ

one-to-one

$$\begin{pmatrix} X_{(n)} - X_{(1)} \\ \frac{X_{(1)} + X_{(n)}}{2} \end{pmatrix}$$ is also a minimal suff. stat. for θ.

6.2.3 Ancillary Statistics

Definition: A statistic $S = S(\underset{\sim}{X})$ is an ancillary statistic if the distribution of S does not depend on θ.

Eg $\quad N(0, \sigma^2) \quad \theta = \sigma^2$

$$\frac{\bar{X}_n}{S_n/\sqrt{n}} \sim t_{n-1} \quad \text{free of } \theta$$