$H_0\ \theta \in \Theta_0$ vs $H_1:\ \theta \in \Theta_1$

$$\lambda(x) = \frac{\sup_{\theta \in \Theta_0} L(\theta \mid \underset{\sim}{x})}{\sup_{\theta \in \Theta_0 \cup \Theta_1} L(\theta \mid \underset{\sim}{x})}$$

$$\frac{\sup_{\Theta_0} L(\theta \mid \underset{\sim}{x})}{\sup_{\Theta_1} L(\theta \mid \underset{\sim}{x})}$$

$H_0:\ \theta \leq \theta_0$ vs $H_1:\ \theta > \theta_0$

Power function.

$$\beta(\theta) = P_\theta(\text{reject } H_0)$$
$$= P_\theta(\underset{\sim}{X} \in RR)$$

$$\phi(\underset{\sim}{X}) = \begin{cases} 1 & \underset{\sim}{X} \in RR \\ 0 & \underset{\sim}{X} \notin RR \end{cases}$$

$$\beta(\theta) = E_\theta[\phi(\underset{\sim}{X})]$$

Type I error : reject H_0 when H_0 is true

$\beta(\theta)$ when $\theta \in \Theta_0$

Type II error : do not reject H_0 when H_1 is true

$1 - \beta(\theta)$ when $\theta \in \Theta_1$

Leval α test :

$$\sup_{\theta \in \Theta_0} \beta(\theta) \leq \alpha$$

Size α test :

$$\sup_{\theta \in \Theta_0} \beta(\theta) = \alpha$$

UMP: uniformly most powerful test.

A test with power function $\beta(\theta)$ is a UMP test.

if $\beta(\theta) \geq \beta^*(\theta)$ for all $\theta \in \Theta_1$

where $\beta^*(\theta)$ is the power function of any other test.

restricted to a class of tests

$\{$ all level α tests $\}$

$H_0: \theta = \theta_0$ versus $H_1: \theta = \theta_1$ $(*)$

simple - versus - simply

denote by $f_{\underset{\sim}{X}}(\underset{\sim}{x} \mid \theta_0)$ the joint pdf of $\underset{\sim}{X} = (X_1, \cdots, X_n)$ at $\theta = \theta_0$

$f_{\underset{\sim}{X}}(\underset{\sim}{x} \mid \theta_1)$ the joint pdf of $\underset{\sim}{X} = (X_1, \cdots, X_n)$ at $\theta = \theta_1$

Let

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \dfrac{f_{\underset{\sim}{X}}(\underset{\sim}{x} \mid \theta_1)}{f_{\underset{\sim}{X}}(\underset{\sim}{x} \mid \theta_0)} > k \\[2ex] 0 & \dfrac{f_{\underset{\sim}{X}}(\underset{\sim}{x} \mid \theta_1)}{f_{\underset{\sim}{X}}(\underset{\sim}{x} \mid \theta_0)} < k \end{cases}$$

for $k \geq 0$, where k is determined by

$$\alpha = P_{\theta_0}(X \in RR) = E_{\theta_0}[\phi(\underset{\sim}{X})]$$

Theorem 8.3.12 (Neyman-Pearson Lemma)

This $\phi(\underset{\sim}{x})$ gives us a (U)MP test for $(*)$

meaning: if you have another test $\phi^*(\underset{\sim}{x})$, where $\underbrace{E_{\theta_0}[\phi^*(\underset{\sim}{x})]}_{\text{Prob of Type I error}} \leq \alpha$,

then $\underbrace{E_{\theta_1}[\phi^*(\underset{\sim}{x})]}_{1 - \text{Prob of Type II error}} \leq E_{\theta_1}[\phi(\underset{\sim}{x})]$

Proof: $E_{\theta_0}[\phi(\underset{\sim}{x})] = \alpha$

for any other level-α $\phi^*(\underset{\sim}{x})$

$$E_{\theta_0}[\phi^*(\underset{\sim}{x})] \leq \alpha$$

$$\boxed{\begin{aligned} &\text{Goal:} \quad E_{\theta_1}[\phi^*(\underset{\sim}{x})] \leq E_{\theta_1}[\phi(\underset{\sim}{x})] \\ &\Leftrightarrow \quad E_{\theta_1}[\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})] \geq 0 \end{aligned}}$$

Define $b(\underset{\sim}{x}) = [\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})][f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_1) - k f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_0)]$

$b(\underset{\sim}{x}) \geq 0$

Case I: $f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_1) - k f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_0) > 0$

$$\phi(\underset{\sim}{x}) = 1 \geq \phi^*(\underset{\sim}{x})$$

$$\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x}) \geq 0$$

Then $b(x) \geq 0$

Case II: $f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_1) - k f_{\underset{\sim}{X}}(x|\theta_0) < 0$

$$\phi(\underset{\sim}{x}) = 0 \leq \phi^*(\underset{\sim}{x})$$

$$\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x}) \leq 0$$

Then $b(\underset{\sim}{x}) \geq 0$

Case III: $f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_1) - k f_{\underset{\sim}{X}}(x|\theta_0) = 0$

Then $b(x) = 0$

$\Rightarrow b(\underset{\sim}{x}) \geq 0$

$$\begin{aligned} b(\underset{\sim}{x}) &= [\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})][f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_1) - k f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_0)] \\ &= [\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})] f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_1) - k[\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})] f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_0) \geq 0 \end{aligned}$$

for $\underset{\sim}{x}$

Then $\int [\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})] f_{\underset{\sim}{x}}(\underset{\sim}{x} | \theta_1)\, d\underset{\sim}{x}$

$$\geq k \int [\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})] f_{\underset{\sim}{x}}(\underset{\sim}{x} | \theta_0)\, d\underset{\sim}{x}$$

$$\underline{E_{\theta_1}[\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})]} = E_{\theta_1}[\phi(\underset{\sim}{x})] - E_{\theta_1}[\phi^*(\underset{\sim}{x})]$$

$$\geq k \underbrace{E_{\theta_0}[\phi(\underset{\sim}{x}) - \phi^*(\underset{\sim}{x})]} \geq 0$$

$$\uparrow$$

$$\underbrace{E_{\theta_0}[\phi(\underset{\sim}{x})]}_{\alpha} - \underbrace{E_{\theta_0}[\phi^*(\underset{\sim}{x})]}_{\leq \alpha} \geq 0 \qquad \#$$

8.3.2 Most powerful tests

Definition: Let $\mathcal{C}$ be a class of tests for testing

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_0^c,$$

where $\Theta_0^c = \Theta \setminus \Theta_0$. A test in $\mathcal{C}$ with power function $\beta(\theta)$ is a **uniformly most powerful (UMP) class $\mathcal{C}$ test** if

$$\beta(\theta) \geq \beta^*(\theta) \ \text{ for all } \theta \in \Theta_0^c,$$

where $\beta^*(\theta)$ is the power function of any other test in $\mathcal{C}$. The "uniformly" part in this definition refers to the fact that the power function $\beta(\theta)$ is larger than (i.e., at least as large as) the power function of any other class $\mathcal{C}$ test **for all** $\theta \in \Theta_0^c$.

Important: In this course, we will restrict attention to tests $\phi(\mathbf{x})$ that are level α tests. That is, we will take

$$\mathcal{C} = \{\text{all level } \alpha \text{ tests}\}.$$

This restriction is analogous to the restriction we made in the "optimal estimation problem" in Chapter 7. Recall that we restricted attention to unbiased estimators first; we then wanted to find the one with the smallest variance (uniformly, for all $\theta \in \Theta$). In the same spirit, we make the same type of restriction here by considering only those tests that are level α tests. This is done so that we can avoid having to consider "silly tests," e.g.,

$$\phi(\mathbf{x}) = 1 \ \text{ for all } \mathbf{x} \in \mathcal{X}.$$

The power function for this test is $\beta(\theta) = 1$, for all $\theta \in \Theta$. This test cannot be beaten in terms of power when H_1 is true! Unfortunately, it is not a very good test when H_0 is true.

Recall: A test $\phi(\mathbf{x})$ with power function $\beta(\theta)$ is a **level** α test if

$$\sup_{\theta \in \Theta_0} \beta(\theta) \leq \alpha.$$

That is, $P(\text{Type I Error}|\theta)$ can be **no larger** than α for all $\theta \in \Theta_0$.

Starting point: We start by considering the **simple-versus-simple** test:

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1.$$

Both H_0 and H_1 specify exactly one probability distribution.

Remark: This type of test is rarely of interest in practice. However, it is the "building block" situation for more interesting problems.

Theorem 8.3.12 (Neyman-Pearson Lemma). Consider testing

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1$$

and denote by $f_{\mathbf{X}}(\mathbf{x}|\theta_0)$ and $f_{\mathbf{X}}(\mathbf{x}|\theta_1)$ the pdfs (pmfs) of $\mathbf{X} = (X_1, X_2, ..., X_n)$ corresponding to θ_0 and θ_1, respectively. Consider the test function

$$\phi(\mathbf{x}) = \begin{cases} 1, & \dfrac{f_{\mathbf{X}}(\mathbf{x}|\theta_1)}{f_{\mathbf{X}}(\mathbf{x}|\theta_0)} > k \\ 0, & \dfrac{f_{\mathbf{X}}(\mathbf{x}|\theta_1)}{f_{\mathbf{X}}(\mathbf{x}|\theta_0)} < k, \end{cases}$$

for $k \geq 0$, where

$$\alpha = P_{\theta_0}(\mathbf{X} \in R) = E_{\theta_0}[\phi(\mathbf{X})]. \tag{8.1}$$

Sufficiency: Any test satisfying the definition of $\phi(\mathbf{x})$ above and Equation (8.1) is a **most powerful** (**MP**) **level** α **test**.

Remarks:

- The necessity part of the Neyman-Pearson (NP) Lemma is less important for our immediate purposes (see CB, pp 388).
- In a simple-versus-simple test, any MP level α test is obviously also UMP level α. Recall that the "uniformly" part in UMP refers to all $\theta \in \Theta_0^c$. However, in a simple H_1, there is only one value of $\theta \in \Theta_0^c$. I choose to distinguish MP from UMP in this situation (whereas the authors of CB do not).

Example 8.13. Suppose that $X_1, X_2, ..., X_n$ are iid beta$(\theta, 1)$, where $\theta > 0$; i.e., the population pdf is

$$f_X(x|\theta) = \theta x^{\theta-1} I(0 < x < 1).$$

Derive the MP level α test for

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta = 2.$$

$\theta_0 = 1$ $\theta_1 = 2$

what if $H_1: \theta = \frac{1}{2}$

If $H_1: \theta = 3$

Solution. The pdf of $\mathbf{X} = (X_1, X_2, ..., X_n)$ is, for $0 < x_i < 1$,

$$f_{\mathbf{X}}(\mathbf{x}|\theta) \stackrel{\text{iid}}{=} \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i \right)^{\theta-1}.$$

Form the ratio

$$\frac{f_{\mathbf{X}}(\mathbf{x}|\theta_1)}{f_{\mathbf{X}}(\mathbf{x}|\theta_0)} = \frac{f_{\mathbf{X}}(\mathbf{x}|2)}{f_{\mathbf{X}}(\mathbf{x}|1)} = \frac{2^n\left(\prod_{i=1}^n x_i\right)^{2-1}}{1^n\left(\prod_{i=1}^n x_i\right)^{1-1}} = 2^n \prod_{i=1}^n x_i$$

$\left(\frac{1}{2}\right)^n \left(\prod_{i=1}^n x_i\right)^{-\frac{1}{2}}$

NP Lemma

The NP Lemma says that the MP level α test uses the rejection rejection

$$R = \left\{\mathbf{x} \in \mathcal{X} : \ 2^n \prod_{i=1}^n x_i > k\right\},$$

$R = \{x : 3^n \prod_{i=1}^n x_i^2 > k\} \iff \{x : \prod x_i > k'\}$

where the constant k satisfies

Distribution hard

$$\alpha = P_{\theta=1}(\mathbf{X} \in R) = P\left(2^n \prod_{i=1}^n X_i > k \,\middle|\, \theta = 1\right).$$

Instead of finding the constant k that satisfies this equation, we rewrite the rejection rule $\{2^n \prod_{i=1}^n x_i > k\}$ in a way that makes our life easier. Note that

$$2^n \prod_{i=1}^n x_i > k \iff \prod_{i=1}^n x_i > \cancel{2^{-n}k} \iff \sum \ln X_i > k$$

$$\iff \sum_{i=1}^n -\ln x_i < -\ln(\cancel{2^{-n}k}) = k', \text{ say.}$$

We have rewritten the rejection rule $\{2^n \prod_{i=1}^n x_i > k\}$ as $\{\sum_{i=1}^n -\ln x_i < k'\}$. Therefore,

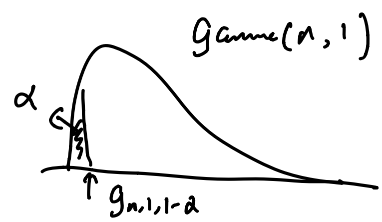
$$\alpha = P\left(2^n \prod_{i=1}^n X_i > k \,\middle|\, \theta = 1\right) = P\left(\sum_{i=1}^n -\ln X_i < k' \,\middle|\, \theta = 1\right).$$

We have now changed the problem to choosing k' to solve this equation above.

Q: Why did we do this?
A: Because it is easier to find the distribution of $\sum_{i=1}^n -\ln X_i$ when $H_0 : \theta = 1$ is true.

Recall that

$$X_i \overset{H_0}{\sim} \mathcal{U}(0,1) \implies -\ln X_i \overset{H_0}{\sim} \text{exponential}(1)$$
$$\implies \sum_{i=1}^n -\ln X_i \overset{H_0}{\sim} \text{gamma}(n,1).$$



Therefore, to satisfy the equation above, we take $k' = g_{n,1,1-\alpha}$, the (lower) α quantile of a gamma$(n,1)$ distribution. This notation for quantiles is consistent with how CB have defined them on pp 386. Thus, the MP level α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ has rejection region

$$R = \left\{\mathbf{x} \in \mathcal{X} : \sum_{i=1}^n -\ln x_i < g_{n,1,1-\alpha}\right\}.$$

→ MP for

$\theta=1$ vs $\theta=2$
$\theta=1$ vs $\theta=3$
$\theta=1$ vs $\theta=4$
⋮

Special case: If $n = 10$ and $\alpha = 0.05$, then $g_{10,1,0.95} \approx 5.425$.

UMP for $H_0 : \theta = 1$ versus $H_1 : \theta > 1$

Q: What is $\beta(2)$, the **power** of this MP test (when $n = 10$ and $\alpha = 0.05$)?
A: We calculate

$$\beta(2) = P\left(\sum_{i=1}^{10} -\ln X_i < 5.425 \,\middle|\, \theta = 2\right).$$

Recall that

$$\begin{aligned} X_i \overset{H_1}{\sim} \text{ beta}(2,1) &\implies -\ln X_i \overset{H_1}{\sim} \text{ exponential}(1/2) \\ &\implies \sum_{i=1}^{10} -\ln X_i \overset{H_1}{\sim} \text{ gamma}(10, 1/2). \end{aligned}$$

Therefore,

$$\beta(2) = \int_0^{5.425} \underbrace{\frac{1}{\Gamma(10)\left(\frac{1}{2}\right)^{10}}\, u^9 e^{-2u}}_{\text{gamma}(10,1/2) \text{ pdf}}\, du \approx 0.643.$$

Proof of NP Lemma. We prove the sufficiency part only. Define the test function

$$\phi(\mathbf{x}) = \begin{cases} 1, & \dfrac{f_{\mathbf{X}}(\mathbf{x}|\theta_1)}{f_{\mathbf{X}}(\mathbf{x}|\theta_0)} > k \\[2ex] 0, & \dfrac{f_{\mathbf{X}}(\mathbf{x}|\theta_1)}{f_{\mathbf{X}}(\mathbf{x}|\theta_0)} < k, \end{cases}$$

where $k \geq 0$ and

$$\alpha = P_{\theta_0}(\mathbf{X} \in R) = E_{\theta_0}[\phi(\mathbf{X})];$$

i.e., $\phi(\mathbf{x})$ is a size α test. We want to show that $\phi(\mathbf{x})$ is MP level α. Therefore, let $\phi^*(\mathbf{x})$ be the test function for any other level α test of H_0 versus H_1. Note that

$$\begin{aligned} E_{\theta_0}[\phi(\mathbf{X})] &= \alpha \\ E_{\theta_0}[\phi^*(\mathbf{X})] &\leq \alpha. \end{aligned}$$

Thus,

$$E_{\theta_0}[\phi(\mathbf{X}) - \phi^*(\mathbf{X})] = \underbrace{E_{\theta_0}[\phi(\mathbf{X})]}_{=\,\alpha} - \underbrace{E_{\theta_0}[\phi^*(\mathbf{X})]}_{\leq\,\alpha} \geq 0.$$

Define

$$b(\mathbf{x}) = [\phi(\mathbf{x}) - \phi^*(\mathbf{x})][f_{\mathbf{X}}(\mathbf{x}|\theta_1) - k f_{\mathbf{X}}(\mathbf{x}|\theta_0)].$$

We want to show that $b(\mathbf{x}) \geq 0$, for all $\mathbf{x} \in \mathcal{X}$.

- **Case 1:** Suppose $f_{\mathbf{X}}(\mathbf{x}|\theta_1) - k f_{\mathbf{X}}(\mathbf{x}|\theta_0) > 0$. Then, by definition, $\phi(\mathbf{x}) = 1$. Because $0 \leq \phi^*(\mathbf{x}) \leq 1$, we have

$$b(\mathbf{x}) = \underbrace{[\phi(\mathbf{x}) - \phi^*(\mathbf{x})]}_{\geq\, 0}\underbrace{[f_{\mathbf{X}}(\mathbf{x}|\theta_1) - k f_{\mathbf{X}}(\mathbf{x}|\theta_0)]}_{>\, 0} \geq 0.$$

- **Case 2:** Suppose $f_{\mathbf{X}}(\mathbf{x}|\theta_1) - kf_{\mathbf{X}}(\mathbf{x}|\theta_0) < 0$. Then, by definition, $\phi(\mathbf{x}) = 0$. Because $0 \leq \phi^*(\mathbf{x}) \leq 1$, we have

$$b(\mathbf{x}) = \underbrace{[\phi(\mathbf{x}) - \phi^*(\mathbf{x})]}_{\leq 0}\underbrace{[f_{\mathbf{X}}(\mathbf{x}|\theta_1) - kf_{\mathbf{X}}(\mathbf{x}|\theta_0)]}_{< 0} \geq 0.$$

- **Case 3:** Suppose $f_{\mathbf{X}}(\mathbf{x}|\theta_1) - kf_{\mathbf{X}}(\mathbf{x}|\theta_0) = 0$. It is then obvious that $b(\mathbf{x}) = 0$.

We have shown that $b(\mathbf{x}) = [\phi(\mathbf{x}) - \phi^*(\mathbf{x})][f_{\mathbf{X}}(\mathbf{x}|\theta_1) - kf_{\mathbf{X}}(\mathbf{x}|\theta_0)] \geq 0$. Therefore,

$$[\phi(\mathbf{x}) - \phi^*(\mathbf{x})]f_{\mathbf{X}}(\mathbf{x}|\theta_1) - k[\phi(\mathbf{x}) - \phi^*(\mathbf{x})]f_{\mathbf{X}}(\mathbf{x}|\theta_0) \geq 0$$
$$\iff [\phi(\mathbf{x}) - \phi^*(\mathbf{x})]f_{\mathbf{X}}(\mathbf{x}|\theta_1) \geq k[\phi(\mathbf{x}) - \phi^*(\mathbf{x})]f_{\mathbf{X}}(\mathbf{x}|\theta_0).$$

Integrating both sides, we get

$$\int_{\mathbb{R}^n} [\phi(\mathbf{x}) - \phi^*(\mathbf{x})]f_{\mathbf{X}}(\mathbf{x}|\theta_1)d\mathbf{x} \geq k\int_{\mathbb{R}^n} [\phi(\mathbf{x}) - \phi^*(\mathbf{x})]f_{\mathbf{X}}(\mathbf{x}|\theta_0)d\mathbf{x},$$

that is,

$$E_{\theta_1}[\phi(\mathbf{X}) - \phi^*(\mathbf{X})] \geq k\underbrace{E_{\theta_0}[\phi(\mathbf{X}) - \phi^*(\mathbf{X})]}_{\geq\ 0,\ \text{shown above}} \geq 0.$$

Therefore, $E_{\theta_1}[\phi(\mathbf{X}) - \phi^*(\mathbf{X})] \geq 0$ and hence $E_{\theta_1}[\phi(\mathbf{X})] \geq E_{\theta_1}[\phi^*(\mathbf{X})]$. This shows that $\phi(\mathbf{x})$ is more powerful than $\phi^*(\mathbf{x})$. Because $\phi^*(\mathbf{x})$ is an arbitrary level α test, we are done. □

Corollary 8.3.13 (NP Lemma with a sufficient statistic T). Consider testing

$$H_0: \theta = \theta_0$$
versus
$$H_1: \theta = \theta_1,$$

$f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta) = h(\underset{\sim}{x})\, g_T(t|\theta)$

and suppose that $T = T(\mathbf{X})$ is a sufficient statistic. Denote by $g_T(t|\theta_0)$ and $g_T(t|\theta_1)$ the pdfs (pmfs) of T corresponding to θ_0 and θ_1, respectively. Consider the test function

$$\phi(t) = \begin{cases} 1, & \dfrac{g_T(t|\theta_1)}{g_T(t|\theta_0)} > k \\[2ex] 0, & \dfrac{g_T(t|\theta_1)}{g_T(t|\theta_0)} < k, \end{cases}$$

for $k \geq 0$, where, with rejection region $S \subset \mathcal{T}$,

$$\alpha = P_{\theta_0}(T \in S) = E_{\theta_0}[\phi(T)].$$

The test that satisfies these specifications is a MP level α test.
Proof. See CB (pp 390).

Implication: In search of a MP test, we can immediately restrict attention to those tests based on a sufficient statistic.

Example 8.14. Suppose $X_1, X_2, ..., X_n$ are iid $\mathcal{N}(\mu, \sigma_0^2)$, where $-\infty < \mu < \infty$ and σ_0^2 is known. Find the MP level α test for

$$H_0 : \mu = \mu_0$$
$$\text{versus}$$
$$H_1 : \mu = \mu_1,$$

where $\mu_1 < \mu_0$.
Solution. The sample mean $T = T(\mathbf{X}) = \overline{X}$ is a sufficient statistic for the $\mathcal{N}(\mu, \sigma_0^2)$ family. Furthermore,

$$T \sim \mathcal{N}\left(\mu, \frac{\sigma_0^2}{n}\right) \quad \Longrightarrow \quad g_T(t|\mu) = \frac{1}{\sqrt{2\pi\sigma_0^2/n}}\, e^{-\frac{n}{2\sigma_0^2}(t-\mu)^2},$$

for $t \in \mathbb{R}$. Form the ratio

$$\frac{g_T(t|\mu_1)}{g_T(t|\mu_0)} = \frac{\frac{1}{\sqrt{2\pi\sigma_0^2/n}}\, e^{-\frac{n}{2\sigma_0^2}(t-\mu_1)^2}}{\frac{1}{\sqrt{2\pi\sigma_0^2/n}}\, e^{-\frac{n}{2\sigma_0^2}(t-\mu_0)^2}} = e^{-\frac{n}{2\sigma_0^2}[(t-\mu_1)^2-(t-\mu_0)^2]}.$$

Corollary 8.3.13 says that the MP level α test rejects H_0 when

what if $\mu_1 > \mu_0$?

$$e^{-\frac{n}{2\sigma_0^2}[(t-\mu_1)^2-(t-\mu_0)^2]} > k \iff t < \frac{2\sigma_0^2 n^{-1}\ln k - (\mu_1^2 - \mu_0^2)}{2(\mu_0 - \mu_1)} = k', \text{ say.}$$

Therefore, the MP level α test uses the rejection region

$$S = \left\{ t \in \mathcal{T} : \frac{g_T(t|\theta_1)}{g_T(t|\theta_0)} > k \right\} = \{t \in \mathcal{T} : t < k'\},$$

where k' satisfies

$$\begin{aligned} \alpha = P_{\mu_0}(T < k') &= P\left(Z < \frac{k' - \mu_0}{\sigma_0/\sqrt{n}}\right) \\ &\Longrightarrow \frac{k' - \mu_0}{\sigma_0/\sqrt{n}} = -z_\alpha \\ &\Longrightarrow k' = \mu_0 - z_\alpha \sigma_0/\sqrt{n}. \end{aligned}$$

Therefore, the MP level α test rejects H_0 when $\overline{X} < \mu_0 - z_\alpha\sigma_0/\sqrt{n}$. This is the same test we would have gotten using $f_{\mathbf{X}}(\mathbf{x}|\mu_0)$ and $f_{\mathbf{X}}(\mathbf{x}|\mu_1)$ with the original version of the NP Lemma (Theorem 8.3.12).

UMP test
for $H_0: \mu = \mu_0$
$H_1: \mu < \mu_0$