From Casella and Berger, do the following problems from Chapter 9:

Homework 10: 2, 3, 4, 12, and 13.
Homework 11: 17, 21, 23, 26, and 36.

These are extra problems that I have given on past exams (in STAT 713 or in related courses). You do not have to turn these in.

9.1. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\theta) = \frac{1}{\theta} x^{(1-\theta)/\theta} I(0 < x < 1),$$

where $\theta > 0$.
(a) Show that $T = T(\mathbf{X}) = \prod_{i=1}^n X_i$ is a sufficient statistic.
(b) Show that

$$Q = Q(T, \theta) = (2/\theta)\ln(1/T)$$

is a pivotal quantity. Use Q to derive a $1 - \alpha$ confidence interval for θ.

9.2. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from the distribution with density

$$f_X(x|\theta) = \frac{\theta}{x^2} I(x \geq \theta),$$

where $\theta > 0$. Construct a $1 - \alpha$ confidence set for θ.

9.3. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from the distribution with density

$$f_X(x|\theta) = a\theta^{-a} x^{a-1} I(0 < x < \theta),$$

where $a \geq 1$ is a known constant and $\theta > 0$ is an unknown parameter.
(a) Show that $T(\mathbf{X}) = X_{(n)}/\theta$ is a pivotal quantity, where $X_{(n)}$ is the maximum order statistic.
(b) Construct a $1 - \alpha$ confidence interval for θ.
(c) Assume that θ has an inverse gamma prior distribution with parameters $\alpha > 0$ and $\beta > 0$ (both known). Write a $1 - \alpha$ credible set for θ. Compare with the confidence interval.

9.4. *Calibration.* Consider the simple linear regression model

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i,$$

for $i = 1, 2, ..., n$, where $\epsilon_i \sim$ iid $\mathcal{N}(0, \sigma^2)$. Suppose that a new value of Y is observed, say, Y_0 (independent of $Y_1, Y_2, ..., Y_n$) and that we wish to estimate the corresponding value of x_0.
(a) Regard x_0 as a parameter. Based on the observed data (x_i, Y_i), $i = 1, 2, ..., n$, and the new value Y_0, find the maximum likelihood estimators of β_0, β_1, x_0, and σ^2.
(b) Derive a $100(1 - \alpha)$ percent confidence interval for x_0. When does such an interval exist? *Hint*: Consider the random variable $V = (Y_0 - \widehat{Y}_0)^2$, where $\widehat{Y}_0 = \widehat{\beta}_0 + \widehat{\beta}_1 x_0$. Find a function of V that has an $F(1, n-2)$ distribution (i.e., a function of V that is pivotal).

9.5. Consider a one-way fixed-effects experiment with $k > 1$ treatments. The following model assumptions are made for the k samples:

$$\begin{array}{ll} \text{Sample 1:} & Y_{11}, Y_{12}, ..., Y_{1n} \sim \text{iid } \mathcal{N}(\mu_1, c_1\sigma^2) \\ \text{Sample 2:} & Y_{21}, Y_{22}, ..., Y_{2n} \sim \text{iid } \mathcal{N}(\mu_2, c_2\sigma^2) \\ \vdots & \\ \text{Sample } k\text{:} & Y_{k1}, Y_{k2}, ..., Y_{kn} \sim \text{iid } \mathcal{N}(\mu_k, c_k\sigma^2). \end{array}$$

The parameters $\mu_1, \mu_2, ..., \mu_k$ and σ^2 are unknown. Note that the design is balanced; i.e., the number of replications n is the same for each treatment. We make the additional assumptions:

- the samples are independent
- the constants $c_1, c_2, ..., c_k$ are known.

(a) Show that

$$\widehat{\sigma}^2 = \frac{1}{k(n-1)} \sum_{i=1}^{k} c_i^{-1} \sum_{j=1}^{n} (Y_{ij} - \overline{Y}_{i+})^2,$$

where $\overline{Y}_{i+} = n^{-1} \sum_{j=1}^{n} Y_{ij}$, is an unbiased estimator of σ^2. Note that if $c_1 = c_2 = \cdots = c_k = 1$, then $\widehat{\sigma}^2$ is simply the mean-squared error (MSE) in the analysis of variance of data from a one-way classification model.
(b) Derive a $1-\alpha$ percent confidence interval for $\theta = a_1\mu_1 + a_2\mu_2 + \cdots + a_k\mu_k$, where $a_1, a_2, ..., a_k$ are constants. *Hint:* Create a pivotal quantity that has a t distribution with $k(n-1)$ degrees of freedom.

9.6. Suppose that $X_1, X_2, ..., X_n$ is an iid sample from

$$f_X(x|\sigma) = \frac{1}{2\sigma} e^{-|x|/\sigma},$$

for $x \in \mathbb{R}$ and $\sigma > 0$. Invert a size α likelihood ratio test (LRT) of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma \neq \sigma_0$ to write a $1 - \alpha$ confidence interval.

9.7. Suppose that $X_1, X_2, ..., X_n$ is an iid sample of $\mathcal{N}(\mu_0, \sigma^2)$ observations, where μ_0 is known and $\sigma^2 > 0$ is unknown. Find as many $1 - \alpha$ interval estimators as possible. Try test inversion, finding a pivot (or pivots), pivoting the cdf of a sufficient statistic, and a Bayesian approach (with an inverse gamma prior, say).

9.8. Suppose $X_1, X_2, ..., X_n$ is an iid Weibull(γ_0, β) sample, where $\gamma_0 > 0$ is known and $\beta > 0$ is unknown.
(a) Find a sufficient statistic for β and create a pivotal quantity from this statistic. Use your pivotal quantity to derive a $1 - \alpha$ interval.
(b) Find the expected length of your interval in part (a). For a fixed sample size n, investigate how you might minimize the expected length of this interval.