HW 1 (Due Feb. 09, 2015)

1. Suppose that $\{W_t\}$ and $\{Z_t\}$ are independent and identically distributed sequences with $P(W_t = 0) = P(W_t = 1) = 0.5$ and $P(Z_t = 1) = P(Z_t = -1) = 0.5$. Define the time series model

$$X_t = W_t(1 - W_{t-1})Z_t.$$

show that $\{X_t\}$ is white but not iid.

2. Verify the following properties of Cov:

   (a) $-1 \leq \rho_{XY} \leq 1$ for any X and Y

   (b) $\text{Cov}(X, X) = \text{Var}(X)$

   (c) $\text{Cov}(X, Y) = \text{Cov}(Y, X)$

   (d) $\text{Cov}(aX, Y) = a\text{Cov}(X, Y)$

   (e) $\text{Cov}(a + X, Y) = \text{Cov}(X, Y)$

   (f) If X and Y are independent, $\text{Cov}(X, Y) = 0$

   (g) $\text{Cov}(X, Y) = 0$ does not imply X and Y are independent

   (h) $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$

   (i) $\text{Cov}(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \text{Cov}(X_i, Y_j)$

3. Prove Theorem 2.1 on Page 12 of the lecture notes

4. For an AR(1) process $\{X_t\}$,

$$X_t = \phi X_{t-1} + W_t,$$

we know the when $|\phi| > 1$, $\{X_t\}$ is not causal (of $\{W_t\}$). However, if we define

$$W_{t*} = X_t - \frac{1}{\phi} X_t,$$

we have a new white noise $\{W_{t*}\}$, and it can be seen that $\{X_t\}$ is now causal of the new white noise $\{W_{t*}\}$. Verify this is a new white noise (check page 30 of lecture notes).

5. Similarly as last problem, check the ARMA(1,1) process. When $|\phi| > 1$ and $|\theta| > 1$, we have non-causal and non-invertible. But if we set

$$\tilde{\phi}(B) = 1 - \phi^{-1}B \quad \text{and} \quad \tilde{\theta}(B) = 1 + \theta^{-1}B$$

and let

$$W_t^* = \tilde{\theta}^{-1}(B)\tilde{\phi}(B)X_t$$

then we have $\{X_t\}$ as a causal and invertible stationary ARMA(1,1) time series. Verify the new $\{W_{t*}\}$ is indeed a white noise (check page 37 of lecture notes).

6. Show when $|\phi| = 1$, there is no stationary solution to

$$X_t - \phi X_{t-1} = W_t + \theta W_{t-1}, \quad W_t \sim \mathrm{WN}(0, \sigma^2), \sigma > 0.$$

7. Consider the AR(2) process defined as the stationary solution to

$$X_t - X_{t-1} + 0.25 X_{t-2} = W_t.$$

   Check the causality of X_t. Find the values of ψ_j such that

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}.$$

   Further find its $\gamma_X(h)$.

8. Consider the MA(2) process defined as the stationary solution to

$$X_t = W_t - W_{t-1} + 0.24 W_{t-2}$$

   Check the invertibility of X_t. Find the values of π_j such that

$$W_t = \sum_{j=-\infty}^{\infty} \pi_j X_{t-j}.$$