STAT810 Probability Theory I

Chapter 1: Sets and Events

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1.1 Introduction

The core classical theorems in probability and statistics are the following:

- *The law of large numbers (LLN):* Suppose $\{X_n : n \geq 1\}$ are independent and identically distributed (iid) random variables with common mean $E(X_n) = \mu$. The LLN says the sample average is approximately equal to the mean, so that

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu.$$

What does the convergence arrow "$\to$" mean?

Suppose $X_i = 1$ if event A occurs, $= 0$ otherwise. Then $n^{-1}\sum_{i=1}^{n} X_i$ is the relative frequency of occurrence of A in n repetitions of the experiment and $\mu = P(A)$. The LLN justifies the frequency interpretation of probabilities and much statistical estimation theory where it underlies the notion of **consistency** of an estimator.

- *Central limit theorem (CLT):* The central limit theorem assures us that sample averages when centered and scaled to have mean 0 and variance 1 have a distribution that is approximately normal. if X_n's are now iid with mean μ and variance σ^2, then

$$P\left(\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \leq x\right) \to N(x) \doteq \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du.$$

This result is "the most" important and most frequently applied result of probability and statistics. How is this result and its variants proved?

1.1 Introduction

- *martingale convergence theorems and optional stopping:* A martingale is a stochastic process $\{X_n : n \geq 0\}$ used to model a fair sequence of gambles. The conditional expectation of your wealth X_{n+1} after the next gamble given the past equals the current wealth X_n; i.e.,

$$E[X_{n+1}|X_0, \ldots, X_n] = X_n.$$

The martingale results on convergence and optimal stopping underlie the modern theory of stochastic processes and are essential tools in application areas such as mathematical finance. What are the basic results and why do they have such far reaching applicability?

1.2 Basic Set Theory

We start by listing some basic notation that are essential to the definition of probability.

- Ω: Sample space (an abstract set representing the sample space of some experiment).
- An individual element of Ω: $\omega \in \Omega$, an outcome of the experiment.
- $\mathcal{P}(\Omega)$: The power set of Ω, that is, the set of all subsets of Ω.
- Subsets $A, B, \ldots$ of Ω: *events*, that is, collections of simple points of Omega.
- Collections of subsets $\mathcal{A}, \mathcal{B}, \ldots$.
- The empty set $\emptyset$.

2.3 Two Constructions

Discrete models: Suppose the sample space $\Omega = \{\omega_1, \omega_2, \dots\}$ is countable. For each i, associate to ω_i the number p_i where

$$\forall i \geq 1, p_i \geq 0, \text{ and } \sum_{i=1}^{\infty} p_i = 1.$$

Define $\mathcal{B} = \mathcal{P}(\Omega)$, the power set, and for $A \in \mathcal{B}$, set

$$P(A) = \sum_{\omega \in A} p_i.$$

Then we have the following properties of P:

(i) $P(A) \geq 0$ for all $A \in \mathcal{B}$.

(ii) $P(\Omega) = \sum_{i=1}^{\infty} p_i = 1$.

(iii) P is σ-additive: if $\{A_n : n \geq 1\}$ are events in $\mathcal{B}$ that are (mutually) disjoint, then

$$P(\cup_{n=1}^{\infty} A_n) = \sum_{\omega_i \in \cup_{n=1}^{\infty} A_n} p_i = \sum_{n=1}^{\infty} \sum_{\omega_i \in A_n} p_i = \sum_{n=1}^{\infty} P(A_n).$$

2.3 Two Constructions

Coin tossing N times: The sample space

$$\Omega = \{0,1\}^N = \{\omega = (\omega_1, \dots, \omega_N) : \omega_i = 0 \text{ or } 1\}.$$

For $p \geq 0$, $q \geq 0$, and $p + q = 1$, define

$$p_\omega = p_{(\omega_1,\dots,\omega_N)} = p^{\sum_{i=1}^N \omega_i} q^{N - \sum_{i=1}^N \omega_i}.$$

Define $B = \mathcal{P}(\Omega)$, for $A \in \mathcal{B}$, define

$$P(A) = \sum_{\omega \in A} p_\omega.$$

This P also satisfies the (i)–(iii) in previous slide.

What if Ω has a continuous feature (uncountable)? e.g., $\Omega = \mathbb{R}$. How to define probabilities for events from Ω? Where to start? Should we use $\mathcal{P}(\Omega)$? See Section 2.4!

Set operations:

1. *Complementation:* The complement of a subset $A \subset \Omega$ is
$$A^c = \{\omega : \omega \notin A\}.$$
2. *Intersection over arbitrary index sets:* Suppose T is some index set and for each $t \in T$ we are given $A_t \subset \Omega$. We define
$$\cap_{t \in T} A_t = \{\omega : \omega \in A_t, \forall t \in T\}.$$
The collection of subsets $\{A_t : t \in T\}$ is *pairwise disjoint* (or *mutually disjoint*) if whenever $t \neq t' \in T$, we have $A_t \cap A_{t'} = \emptyset$.
3. *Union over arbitrary index sets:*
$$\cup_{t \in T} A_t = \{\omega : \omega \in A_t, \text{ for some } t \in T\}.$$

Set operations:

1. *Set difference:* $A \setminus B = A \cap B^c$.
2. *Symmetric different:* $A \triangle B = (A \setminus B) \cup (B \setminus A) = A^c \triangle B^c$.

Simple relations between sets:

- *Containment:* $A \subset B$ iff $A \cap B = A$ or iff $\omega \in A$ implies $\omega \in B$.
- *Equality:* $A = B$ iff $A \subset B$ and $B \subset A$ or iff $\omega \in A \iff \omega \in B$.

Example 1.2.1

$$
\begin{aligned}
\cup_{n=1}^{\infty}[0, n/(n+1)) =& [0,1) \\
\cap_{n=1}^{\infty}(0, 1/n) =& \emptyset \\
\cup_{n=1}^{\infty}[1/n, (n+1)/n) =& (0,2) \\
\cap_{n=1}^{\infty}[1/n, (n+1)/n) =& \{1\} \\
\cap_{n=1}^{\infty}[0, 1/n) =& 0.
\end{aligned}
$$

Some laws: (HW 1-1: prove these laws.)

1. *Complementation:* $(A^c)^c = A$, $\emptyset^c = \Omega$, $\Omega^c = \emptyset$.
2. *Commutativity:* $A \cap B = B \cap A$, $A \cup B = B \cup A$.
3. *Associativity:*

$$(A \cup B) \cup C = A \cup (B \cup C),$$
$$(A \cap B) \cap C = A \cap (B \cap C).$$

4. *De Morgan's laws:* Let T be an index set,

$$(\cup_{t \in T} A_t)^c = \cap_{t \in T} A_t^c, \quad (\cap_{t \in T} A_t)^c = \cup_{t \in T} A_t^c.$$

5. *Distributivity:*

$$B \cap (\cup_{t \in T} A_t) = \cup_{t \in T} (B \cap A_t),$$
$$B \cup (\cap_{t \in T} A_t) = \cap_{t \in T} (B \cup A_t).$$

1.2.1 *Powerful* Indicator Functions

If $A \subset \Omega$, we define the indicator function of A as

$$\boldsymbol{I}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \in A^c. \end{cases}$$

Some simple properties:

- $\boldsymbol{I}_A \leq \boldsymbol{I}_B \iff A \subset B$
- $\boldsymbol{I}_{A^c} = 1 - \boldsymbol{I}_A$
- $\boldsymbol{I}_{A \cup B} = \max(\boldsymbol{I}_A, \boldsymbol{I}_B)$
- $\boldsymbol{I}_{A \cap B} = \min(\boldsymbol{I}_A, \boldsymbol{I}_B)$
- $\boldsymbol{I}_{A \triangle B} = \boldsymbol{I}_A + \boldsymbol{I}_B \pmod 2$
- $E[\boldsymbol{I}_A(X)] = P(X \in A)$, where X is a well-defined random variable.

1.3 Limits of Sets

For a sequence of **real numbers** a_n, we have

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 1} \inf_{k\geq n} a_k, \quad \limsup_{n\to\infty} a_n = \inf_{n\geq 1} \sup_{k\geq n} a_k,$$

and $\lim_{n\to\infty} a_n = a$ exists iff $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a$.

Similarly, for a sequence of **sets** $\{A_n \subset \Omega : n \geq 1\}$, we define

$$\inf_{k\geq n} A_k = \cap_{k=n}^{\infty} A_k, \quad \sup_{k\geq n} A_k = \cup_{k=n}^{\infty} A_k,$$

$$\liminf_{n\to\infty} A_n = \sup_{n\geq 1} \inf_{k\geq n} A_k = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k,$$

$$\limsup_{n\to\infty} A_n = \inf_{n\geq 1} \sup_{k\geq n} A_k = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k.$$

The **limit** of a set sequence $\{A_n\}$ exists, denoted by A, if

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A,$$

in shorthand, $\lim_{n\to\infty} A_n = A$ or $A_n \to A$.

Remark (a)

if $\omega \in \limsup_{n\to\infty} A_n$, then for every n, $\omega \in \cup_{k\geq n} A_k$; i.e., there exists some $k_n \geq n$ such that $\omega \in A_{k_n}$. Therefore

$$\sum_{j=1}^{\infty} \boldsymbol{I}_{A_j}(\omega) \geq \sum_{n=1}^{\infty} \boldsymbol{I}_{A_{k_n}}(\omega) = \infty.$$

Conversely, if $\sum_{j=1}^{\infty} \boldsymbol{I}_{A_j}(\omega) = \infty$, then there exists $k_n \to \infty$ such that $\omega \in A_{k_n}$, and therefore for all n, $\omega \in \cup_{j\geq n} A_j$; i.e., $\omega \in \limsup_{n\to\infty} A_n$.

Lemma 1.3.1

Let $\{A_n\}$ be a sequence of subsets of Ω.

(a) For $\limsup$ we have the interpretation

$$\limsup_{n\to\infty} A_n = \left\{\omega : \sum_{n=1}^{\infty} \boldsymbol{I}_{A_n}(\omega) = \infty\right\}$$
$$= \{\omega : \omega \in A_{n_k}, k = 1, 2, \ldots\}$$

for some subsequence n_k depending on ω. Consequently, we write

$$\limsup_{n\to\infty} A_n = [A_n \ i.o.]$$

where $i.o.$ stands for infinitely often.

Remark (b)

if $\omega \in \liminf_{n\to\infty} A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$, then there exists an n such that $\omega \in \cap_{k\geq n} A_k$; Therefore

$$\sum_{n=1}^{\infty} \boldsymbol{I}_{A_n^c}(\omega) \leq \sum_{j=1}^{n-1} \boldsymbol{I}_{A_j^c}(\omega) < \infty.$$

Conversely, if $\sum_{n=1}^{\infty} \boldsymbol{I}_{A_n^c}(\omega) < \infty$, then there exists n_0 such that $\boldsymbol{I}_{A_n^c}(\omega) = 0$ for $n \geq n_0$; i.e., $\omega \in \liminf_{n\to\infty} A_n$.

Lemma 1.3.1

(b) For $\liminf$ we have the interpretation

$$\begin{aligned}\liminf_{n\to\infty} A_n =& \{\omega : \omega \in A_n \text{ for all } n \text{ except a finite number}\} \\ =& \left\{\omega : \sum_{n=1}^{\infty} \boldsymbol{I}_{A_n^c}(\omega) < \infty\right\} = \{\omega : \omega \in A_n, \forall n \geq n_0(\omega)\}.\end{aligned}$$

In addition, we have

$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n,$$

$$(\liminf_{n\to\infty} A_n)^c = (\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k)^c = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k^c = \limsup_{n\to\infty} A_n^c.$$

Example 1.3.1

$\liminf_{n\to\infty}[0, n/(n+1)) = \limsup_{n\to\infty}[0, n/(n+1)) = [0, 1)$.

Say $A_n = B$ if n is odd; $= C$ otherwise. What are $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$?

Suppose $a_n > 0$, $b_n > 1$ and $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 1$. Define $A_n = [a_n, b_n)$, what are $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$?

For a sequence of random variables $\{X_n\}$, $X_n \to X_0$ **almost surely** iff

$$P\{\omega : lim_{n\to\infty} X_n(\omega) = X_0(\omega)\} = 1.$$

A criterion for this is that for all $\epsilon > 0$

$$P([|X_n - X_0| > \epsilon]\ i.o.) = 0.$$

That is, with $A_n = [|X_n - X_0| > \epsilon]$, we need to check

$$P\left(\limsup_{n\to\infty} A_n\right) = 0.$$

A sufficient condition (**Borel-Cantelli Lemma**) for this to hold is

$$\sum_{n=1}^{\infty} P(A_n) < \infty.$$

1.4 Monotone Sequences

Just like monotone sequences of real numbers, the limit of a monotone sequence of sets always exists.

Definition

$\{A_n\}$ is monotone-increasing, $A_n \uparrow$, if $A_1 \subset A_2 \subset$, then

$$\lim_{n\to\infty} A_n = \cup_{n=1}^{\infty} A_n.$$

$\{A_n\}$ is monotone-decreasing, $A_n \downarrow$, if $A_1 \supset A_2 \supset$, then

$$\lim_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n.$$

Consequently, since for any sequences B_n, we have $\inf_{k\geq n} B_k \uparrow$ and $\sup_{k\geq n} B_k \downarrow$, it follows that

$$\liminf_{n\to\infty} B_n = \lim_{n\to\infty}\left(\inf_{k\geq n} B_k\right), \quad \limsup_{n\to\infty} B_n = \lim_{n\to\infty}\left(\sup_{k\geq n} B_k\right)$$

Example 1.4.1

1. Let $0 \leq a_n < \infty$ be a sequence of numbers. Then, (yes or no),

$$\sup_{n\geq 1}[0, a_n) = [0, \sup_{n\geq 1} a_n), \quad \sup_{n\geq 1}[0, a_n] = [0, \sup_{n\geq 1} a_n].$$

2. How about

$$\lim_{n\to\infty}[0, 1-1/n] = [0,1) = \lim_{n\to\infty}[0, 1-1/n)$$
$$\lim_{n\to\infty}[0, 1+1/n] = [0,1] = \lim_{n\to\infty}[0, 1+1/n)?$$

3. Suppose $a_n \downarrow a$, does $(-\infty, a_n] \downarrow (-\infty, a]$?
Or $a_n \uparrow a$, does $(-\infty, a_n] \uparrow (-\infty, a]$?

Addition parallels between sets and functions:

(1) $\boldsymbol{I}_{\inf_{n\geq k} A_n} = \inf_{n\geq k} \boldsymbol{I}_{A_n}$ and $\boldsymbol{I}_{\sup_{n\geq k} A_n} = \sup_{n\geq k} \boldsymbol{I}_{A_n}$.

(2) $\boldsymbol{I}_{\cup_n A_n} \leq \sum_n \boldsymbol{I}_{A_n}$ where equality holds if A_n's are mutually disjoint.

(3) $\boldsymbol{I}_{\limsup_{n\to\infty} A_n} = \limsup_{n\to\infty} \boldsymbol{I}_{A_n}$ and $\boldsymbol{I}_{\liminf_{n\to\infty} A_n} = \liminf_{n\to\infty} \boldsymbol{I}_{A_n}$.

(4) $A_n \to A$ iff $\boldsymbol{I}_{A_n} \to \boldsymbol{I}_A$.

We note that (4) can be implied by (3), (3) can be implied by (1), and (2) is trivial. To prove (1),

$$\begin{aligned} \boldsymbol{I}_{\inf_{n\geq k}} A_n = 1 &\text{ iff } \omega \in \inf_{n\geq k} A_n \\ &\text{iff } \omega \in A_n \text{ for all } n \geq k \\ &\text{iff } \inf_{n\geq k} \boldsymbol{I}_{A_n} = 1. \end{aligned}$$

1.5 Set Operations and Closure

The class of real numbers is closed under addition and multiplication. Can a class of set be closed under certain set operations? Suppose $\mathcal{C} \subset \mathcal{P}(\Omega)$ is a collection of subsets of Ω.

(1) Arbitrary union: Let T be any arbitrary index set (could be finite, countable, or a subset of the real line) and assume for $t \in T$, $A_t \in \mathcal{C}$, the arbitrary union is $\cup_{t \in T} A_t$.

(2) Countable union: $\cup_{j=1}^{\infty} A_j$.

(3) Finite union: $\cup_{j=1}^{n} A_j$.

(4) Arbitrary intersection: $\cap_{t \in T} A_t$.

(5) Countable intersection: $\cap_{j=1}^{\infty} A_j$.

(6) Finite intersection: $\cap_{j=1}^{n} A_j$.

(7) Complementation: A^c

(8) Monotone limits: $\lim_{n\to\infty} A_n$ where $\{A_n\}$ is a monotone sequence of sets in $\mathcal{C}$.

1.5 Set Operations and Closure

Definition 1.5.1 (Closure)

Let $\mathcal{C}$ be a collection of subsets of Ω. The $\mathcal{C}$ is **closed** under one of the set operations (1)–(8) listed above if the set obtained by performing the set operation on sets $\mathcal{C}$ yields a set in $\mathcal{C}$.

Example 1.5.1

1. Suppose $\Omega = \mathbb{R}$, and $\mathcal{C} = \{(a, b] : -\infty < a \leq b < \infty\}$. Then $\mathcal{C}$ is not closed under finite unions but is closed under finite intersections.
2. Suppose $\Omega = R$ and $\mathcal{C}$ consists of the open subsets of $\mathbb{R}$. Then $\mathcal{C}$ is not closed under complementation.

Closure is important for us to assign probabilities. In general, we cannot assign probabilities to all subsets (events). We combine and manipulate events to make more complex events via set operations. We require that certain set operations do not carry events outside the event space.

Definition 1.5.2

An **algebra** (or a **field**) is a non-empty class of subsets of Ω **closed** under finite union, finite intersection, and complementation; i.e., For $\mathcal{A}$ to be an algebra as long as

(i) $\Omega \in \mathcal{A}$

(ii) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$

(iii) $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$ (or by De Morgan's laws, $A \cap B \in \mathcal{A}$).

Definition 1.5.3

A σ-**algebra** (or a σ-**field**) is a non-empty class of subsets of Ω **closed** under countable union, countable intersection, and complementation; i.e., For $\mathcal{B}$ to be a σ-algebra as long as

(i) $\Omega \in \mathcal{B}$

(ii) $B \in \mathcal{B}$ implies $B^c \in \mathcal{B}$

(iii) $B_i \in \mathcal{B}$, $i \geq 1$ implies $\cup_{i=1}^{\infty} B_i \in \mathcal{B}$ (or $\cap_{i=1}^{\infty} B_i \in \mathcal{B}$).

1.5.1 Examples

(1) What is the most simple σ-algebra?

(2) The power set $\mathcal{B} = \mathcal{P}(\Omega)$ is a σ-algebra.

(3) The countable/co-countable σ-algebra. Let $\Omega = \mathbb{R}$ and

$$\mathcal{B} = \{A \subset \mathbb{R} : A \text{ is countable}\} \cup \{A \subset \mathbb{R} : A^c \text{ is countable}\}.$$

$\mathcal{B}$ is a σ-algebra. (Is $\mathcal{B}$ closed under arbitrary union?)

(4) Let $\Omega = (0, 1]$ and $\mathcal{A}$ consists of the empty set and all finite unions of disjoint intervals of the form $(a, a']$, $0 \leq a \leq a' \leq 1$. Is $\mathcal{A}$ an algebra? Is it a σ-algebra?

(5) Let $\Omega = \mathbb{N}$, the integers. Define $\mathcal{A} = \{A \subset \mathbb{N} : A \text{ or } A^c \text{ is finite}\}$. Is $\mathcal{A}$ an algebra? Is it a σ-algebra?

(6) Suppose $\Omega = \{e^{i2\pi\theta} : 0 \leq \theta < 1\}$ is the unit circle. Let $\mathcal{A}$ be the collection of arcs on the unit circle with rational endpoints. Is $\mathcal{A}$ an algebra? Is it a σ-algebra?

(7) Find two σ-algebras, the union of which is not an algebra.

1.6 The σ-algebra Generated by a Given Class $\mathcal{C}$

Definition 1.6.1

Let $\mathcal{C}$ be a collection of subsets of Ω. The σ-algebra generated by $\mathcal{C}$, denoted by $\sigma(\mathcal{C})$, is a σ-algebra satisfying

(a) $\sigma(\mathcal{C}) \supset \mathcal{C}$

(b) if $\mathcal{B}$ is some other σ-algebra containing $\mathcal{C}$, then $\mathcal{B} \supset \sigma(\mathcal{C})$.

The $\sigma(\mathcal{C})$ is known as the minimal σ-algebra over $\mathcal{C}$.

Proposition 1.6.1 Uniqueness (a non-constructive statement)

Given a class $\mathcal{C}$ of subsets of Ω, there is a unique minimal σ-algebra containing $\mathcal{C}$, which is

$$\cap_{\mathcal{B} \in \mathcal{N}} \mathcal{B}, \text{ where } \mathcal{N} = \{\mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{-algebra}, \mathcal{B} \supset \mathcal{C}\}.$$

Example

1. If $\mathcal{A} = A$, a single set, then $\sigma(\mathcal{A}) = \{\emptyset, A, A^c, \Omega\}$.
2. If $\mathcal{A}$ is a σ-algebra already, then $\sigma(\mathcal{A}) = \mathcal{A}$.

2.2 More on Closure

Let $\mathcal{A}$ be a collection of subsets of Ω

- $\mathcal{A}$ is a monotone class if $\lim_{n\to\infty} A_n \in \mathcal{A}$ where $\{A_n\}$ is a monotone sequence of sets in $\mathcal{A}$.
- $\mathcal{A}$ is a π-system if $A, B \in \mathcal{A}$ implies $A \cap B \in \mathcal{A}$.
- $\mathcal{A}$ is a Dynkin system (also known as λ-system) if
  - (i) $\Omega \in \mathcal{A}$
  - (ii) $A, B \in \mathcal{A}$ and $B \subset A$ imply $A \setminus B \in \mathcal{A}$
  - (iii) $A_n \in \mathcal{A}$ and $A_n \uparrow$ imply $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$.

  or equivalently, if
  - (i) $\Omega \in \mathcal{A}$
  - (ii) A implies $A^c \in \mathcal{A}$
  - (iii) $A_n \in \mathcal{A}$ and $A_n \cap A_m = \emptyset$ for $n \neq m$ imply $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$.

2.2 More on Closure

Theorem (HW 1-2: Prove this Theorem)

1. Every algebra is a π-system.
2. Every σ-algebra is an algebra.
3. An algebra is a σ-algebra iff it is a monotone class.
4. Every σ-algebra is a Dynkin system.
5. A Dynkin system is a σ-algebra iff it is a π-system.
6. Every Dynkin system is a monotone class.
7. Every σ-algebra is a monotone class.
8. The power set of any subset of Ω is a σ-algebra on that subset.
9. The intersection of any number of σ-algebra, countable or uncountable, is again, a σ-algebra.
10. If Ω and Ω' are sets, $\mathcal{A}'$ a σ-algebra in Ω' and $T : \Omega \to \Omega'$ a mapping, then $T^{-1}(\mathcal{A}') = \{T^{-1}(A') : A' \in \mathcal{A}'\}$ is a σ-algebra on Ω (Proposition 3.1.1).

2.2 More on Closure (Other Generators)

In previous slide, we focus on the structure σ-algebra. We also have other structures; e.g., the monotone-class, the π-system, the Dynkin system. Now we fix a structure in mind. Call it $\mathcal{S}$. Then we can make the following definition.

Definition 2.2.1

The (minimal) structure $\mathcal{S}$ generated by a class $\mathcal{C}$ is a non-empty structure satisfying

(i) $\mathcal{S} \supset \mathcal{C}$,

(ii) If $\mathcal{S}'$ is some other structure containing $\mathcal{C}$, then $\mathcal{S}' \supset \mathcal{S}$.

We denote the minimal structure by $\mathcal{S}(\mathcal{C})$. If $\mathcal{S}$ is the σ-algebra, then $\mathcal{S}(\mathcal{C}) = \sigma(\mathcal{C})$. The Dynkin system generated by $\mathcal{A}$ is $\mathcal{L}(\mathcal{A})$, the monotone class generated by $\mathcal{A}$ is $\mathcal{M}(\mathcal{A})$.

Proposition 2.2.1

The minimal structure $\mathcal{S}$ exists and is unique as $\mathcal{S}(\mathcal{C}) = \cap_{\mathcal{G} \in \mathcal{N}} \mathcal{G}$ where $\mathcal{N} = \{\mathcal{G} : \mathcal{G} \text{ is a structure}, \mathcal{G} \supset \mathcal{C}\}$.

Some connections

1. Let $\mathcal{A}$ be an algebra, then $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$.
2. If $\mathcal{A}$ is an algebra and $\mathcal{G}$ a monotone class containing $\mathcal{A}$, then $\sigma(\mathcal{A}) \subset \mathcal{G}$.
3. If $\mathcal{A}$ is a σ-algebra, then $\mathcal{M}(\mathcal{A}) = \mathcal{L}(\mathcal{A}) = \sigma(\mathcal{A}) = \mathcal{A}$.

Theorem 2.2.2 (Dynkin's theorem)

(a) If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a Dynkin system such that $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.

(b) If $\mathcal{P}$ is a π-system, $\sigma(\mathcal{P}) = \mathcal{L}(\mathcal{P})$.

For example, $\mathcal{P} = \{(-\infty, x] : x \in \mathbb{R}\}$ is a π-system (will be used late to justify the uniqueness of a cdf function when defining probability).

1.7 Borel Sets on the Real Line

Suppose $\Omega = \mathbb{R}$ and let

$$\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}.$$

Define

$$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$$

and call $\mathcal{B}(\mathbb{R})$ the **Borel** subsets of $\mathbb{R}$. Equivalently, we have

$$\begin{aligned}
\mathcal{B}(\mathbb{R}) &= \sigma(\{(a, b] : -\infty \le a \le b < \infty\}) \\
&= \sigma(\{(a, b) : -\infty \le a \le b \le \infty\}) \\
&= \sigma(\{[a, b) : -\infty < a \le b \le \infty\}) \\
&= \sigma(\{[a, b] : -\infty < a \le b < \infty\}) \\
&= \sigma(\{(-\infty, x) : x \in \mathbb{R}\});
\end{aligned}$$

i.e., we can generate the Borel sets with any kind of interval: open, closed, semi-open, finite, semi-finite, etc. Proofs can be done using this trick: $(a, b) = \cup_{n=1}^{\infty}(a, b - 1/n]$ and $(a, b] = \cap_{n=1}^{\infty}(a, b + 1/n)$.

1.7 Borel Sets on the Real Line

In fact

$$\mathcal{B}(\mathbb{R}) = \sigma(\text{open subsets of } \mathbb{R}).$$

(Why? because any open subset of $\mathbb{R}$ can be written as a countable union of disjoint open intervals)

In general, if $\mathbb{E}$ is a metric space, it is ususal to define $\mathcal{B}(\mathbb{E})$, the σ-algebra on $\mathbb{E}$, to be the σ-algebra generated by the open subsets of $\mathbb{E}$. Then $\mathcal{B}(\mathbb{E})$ is call the **Borel** σ-algebra. Examples of metric spaces $\mathbb{E}$ that are useful to consider are

- $\mathbb{R}$
- $\mathbb{R}^d$
- $\mathbb{R}^\infty$, the space of all real sequences
- $C[0, \infty)$, the space of continuous functions on $[0, \infty)$.

Theorem 1.8.1 (HW 1-3: prove this theorem)

Let $\Omega_0 \subset \Omega$.

(1) If $\mathcal{B}$ is a σ-algebra of subsets of Ω, then $\mathcal{B}_0 = \{A \cap \Omega_0 : A \in \mathcal{B}\}$ is a σ-algebra of subsets of Ω_0.

(2) Suppose $\mathcal{C}$ is a class of subsets of Ω and $\mathcal{B} = \sigma(\mathcal{C})$. Set

$$\mathcal{C}_0 = \mathcal{C} \cap \Omega_0 = \{A \cap \Omega_0 : A \in \mathcal{C}\}.$$

Then

$$\sigma(\mathcal{C}_0) = \sigma(\mathcal{C} \cap \Omega_0) = \sigma(\mathcal{C}) \cap \Omega_0$$

in Ω_0.

For example, if we take $\Omega = \mathbb{R}$ and $\Omega_0 = (0, 1]$. Then

$$\mathcal{B}(0, 1] = \sigma(\text{subintervals of } (0, 1]) = \mathcal{B}(\mathbb{R}) \cap (0, 1].$$

(Other HW 1 problems: Section 1.9, Q8–11, Q14-15, Q17-20, Q26-29, Q31, Q34-35, Q44)