STAT 810 Probability Theory I

Chapter 2: Probability Spaces

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A **probability space** is a triple $(\Omega, \mathcal{B}, P)$ where

- Ω is the sample space corresponding to outcomes of some experiment.
- $\mathcal{B}$ is the σ-algebra of subsets of Ω. These subsets are called **events**.
- P is a probability measure; that is, P is a function with domain $\mathcal{B}$ and range $[0, 1]$ such that (**Kolmogorov axioms**)
  (i) $P(A) \geq 0$ for all $A \in \mathcal{B}$.
  (ii) P is σ-additive: if $\{A_n : n \geq 1\}$ are events in $\mathcal{B}$ that are disjoint, then
  $$P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n).$$
  (iii) $P(\Omega) = 1$.

1. $P(A^c) = 1 - P(A)$.
2. $P(\emptyset) = 0$.
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
4. The *inclusion-exclusion* formula: $P(\cup_{j=1}^n A_j) = \sum_{j=1}^n P(A_j) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^n P(A_1 \cap \cdots \cap A_n)$.
5. *Bonferroni inequalities*: $P(\cup_{j=1}^n A_j) \le \sum_{j=1}^n P(A_j)$, or $P(\cup_{j=1}^n A_j) \ge \sum_{j=1}^n P(A_j) - \sum_{1 \le i < j \le n} P(A_i \cap A_j)$.
6. The *monotonicity property*: if $A \subset B$, then $P(A) \le P(B)$.
7. *Subadditivity*: $P(\cup_{n=1}^\infty A_n) \le \sum_{n=1}^\infty P(A_n)$.
8. *Continuity*: The measure P is continuous in the sense that if $A_n \uparrow A$, then $P(A_n) \uparrow P(A)$; if $A_n \downarrow A$, then $P(A_n) \downarrow P(A)$; if $A_n \to A$, then $P(A_n) \to P(A)$.
9. *Fatou's lemma*: $P(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} P(A_n) \le \limsup_{n\to\infty} P(A_n) \le P(\limsup_{n\to\infty} A_n)$.

Proof of continuity: $A_n \uparrow A$. Define $B_n = A_n \setminus A_{n-1}$. Then $\{B_n\}$ is a disjoint sequence of events such that $\cup_{t=1}^{n} B_t = A_n$ and $\cup_{t=1}^{\infty} B_t = A$. Then $P(A) = P(\cup_{t=1}^{\infty} B_t) = \sum_{i=1}^{\infty} P(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) = \lim_{n\to\infty} P(\cup_{i=1}^{n} B_i) = \lim_{n\to\infty} P(A_n)$.

Proof of Fatou's lemma:
$P(\liminf_{n\to\infty} A_n) = P(\cup_{n=1}^{\infty} \cap_{k\ge n} A_k) = P(\lim_{n\to\infty} \cap_{k\ge n} A_k) = \lim_{n\to\infty} P(\cap_{k\ge n} A_k) = \liminf_{n\to\infty} P(\cap_{k\ge n} A_k) \le \liminf_{n\to\infty} P(A_n)$.
Likewise
$P(\limsup_{n\to\infty} A_n) = P(\cap_{n=1}^{\infty} \cup_{k\ge n} A_k) = P(\lim_{n\to\infty} \cup_{k\ge n} A_k) = \lim_{n\to\infty} P(\cup_{k\ge n} A_k) = \limsup_{n\to\infty} P(\cup_{k\ge n} A_k) \ge \limsup_{n\to\infty} P(A_n)$.

Example 2.1.1 (HW 2-1: Prove (i)–(iii).)
Let $\Omega = \mathbb{R}$, and suppose P is a probability measure on $\mathbb{R}$. Define

$$F(x) = P((-\infty, x]), x \in \mathbb{R}.$$

Then

(i) F is right continuous,

(ii) F is monotone non-decreasing,

(iii) F has limits at $\pm\infty$: $F(\infty) = \lim_{x\uparrow\infty} F(x) = 1$ and $F(-\infty) = \lim_{x\downarrow-\infty} F(x) = 0$.

Definition 2.1.1 A function $F : \mathbb{R} \to [0, 1]$ satifying (i),(ii),(iii) is called a (probability) (cumulative) distribution function, (in shorthand, df or cdf).

A probability measure on $\mathbb{R}$ is uniquely determined by its distribution function.

Corollary 2.2.2

Let $\Omega = \mathbb{R}$. Let P_1, P_2 be two probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that their dfs are equal:

$$\forall x \in \mathbb{R} : F_1(x) = P_1((-\infty, x]) = F_2(x) = P_2((-\infty, x]).$$

Then $P_1 = P_2$ on $\mathcal{B}(\mathbb{R})$; i.e., $\forall A \in \mathcal{B}(\mathbb{R})$, $P_1(A) = P_2(A)$.

We note $\mathcal{P} = \{(-\infty, x] : x \in \mathbb{R}\}$ is a π-system and $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$.

More general: Corollary 2.2.1

If P_1, P_2 are two probability measures on $(\Omega, \mathcal{B})$ and if $\mathcal{P}$ is a π-system such that $\forall A \in \mathcal{P}$, $P_1(A) = P_2(A)$, then $\forall B \in \sigma(\mathcal{P})$, $P_1(B) = P_2(B)$.

Proof of Corollary 2.2.1 follows

Proposition 2.2.3

Let P_1, P_2 be two probability measures on $(\Omega, \mathcal{B})$. The class $\mathcal{L} = \{A \in \mathcal{B} : P_1(A) = P_2(A)\}$ is a Dynkin system.

and

(a) in Theorem 2.2.2 (Dynkin's theorem)

If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a Dynkin system such that $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.

2.4 Constructions of Probability Spaces

How to construct probability spaces when Ω is uncountable? Same as in the countable case, we start with a simple class of subsets $\mathcal{S}$ of Ω to which the assignment of probabilities is obvious or natural. Then we extend this assignment of probabilities to $\sigma(\mathcal{S})$. Herein, we focus on the case where $\Omega = \mathbb{R}$. Suppose we are given a distribution function F, we could take $\mathcal{S}$ to be

$$\mathcal{S} = \{(x, y] : -\infty \leq x \leq y \leq \infty\}$$

and then define P on $\mathcal{S}$ to be

$$P((x, y]) = F(y) - F(x).$$

The problem is to extend the definition of P from $\mathcal{S}$ to $\sigma(\mathcal{S}) = \mathcal{B}(\mathbb{R})$, the Borel sets.

*What do we mean by **extension**?* Suppose two classes $\mathcal{G}_1$, $\mathcal{G}_2$ of subsets of Ω such that $\mathcal{G}_1 \subset \mathcal{G}_2$ and two set functions

$$P_i : \mathcal{G}_i \mapsto [0,1], i = 1, 2,$$

we say P_2 is an **extension** of P_1 (or P_1 extends to P_2) if P_2 restricted to $\mathcal{G}_1$ equals P_1; i.e., $P_2(A_1) = P_1(A_1)$ for all $A_1 \in \mathcal{G}_1$.

Semi-algebra

A class $\mathcal{S}$ of subsets of Ω is a **semi-algebra** if the following holds

(i) $\emptyset, \Omega \in \mathcal{S}$.

(ii) $\mathcal{S}$ is a π-system.

(iii) If $A \in \mathcal{S}$, then there exist some finite n and disjoint sets $C_1, \ldots, C_n$ with each $C_i \in \mathcal{S}$ such that $A^c = \cup_{i=1}^n C_i$.

$\mathcal{S} = \{(x, y] : -\infty \leq x \leq y \leq \infty\}$ is a semialgrebra (not an algebra).

2.4 Constructions of Probability Spaces

The following Theorem show that, we can start with assigning probabilities to $\mathcal{S}$, then extend it to $\sigma(\mathcal{S})$ uniquely!

Theorem 2.4.3 (Extension Theorem)

Suppose $\mathcal{S}$ is a *semi-algebra* of subsets of Ω and that P is a σ-additive set function mapping $\mathcal{S}$ into $[0, 1]$ such that $P(\Omega) = 1$. There is a unique probability measure on $\sigma(\mathcal{S})$ that extends P.

Recall that P σ-additive if

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{n} P(A_i)$$

holds for mutually disjoint $\{A_n\}$ with $A_j \in \mathcal{G}$ and $\cup_{j=1}^{\infty} A_j \in \mathcal{G}$.

2.4 Constructions of Probability Spaces

We start with something we have known:

Lebesgue Measure on $(0, 1]$

Suppose $\Omega = (0, 1]$, $\mathcal{B} = \mathcal{B}((0, 1])$, $\mathcal{S} = \{(a, b] : 0 \leq a \leq b \leq 1\}$.
Define on $\mathcal{S}$ the function $\lambda : \mathcal{S} \mapsto [0, 1]$ by

$$\lambda(\emptyset) = 0, \quad \lambda(a, b] = b - a.$$

Based on the Extension Theorem, we can extend λ to $\sigma(\mathcal{S}) = \mathcal{B}$.
The extended measure is the Lebesgue Measure (see Section 2.5.1).

2.5 Probability Construction

Now we discuss the construction of a probability measure on $\mathbb{R}$ with a given df $F(x)$. Based on $F(x)$, we can construct a probability measure on $\mathbb{R}$, P_F (see Section 2.5.2), such that

$$P_F((x, y]) = \underbrace{F(y)}_{b} - \underbrace{F(x)}_{a} = \lambda(a, b].$$

Intuition: For $A \subset \mathbb{R}$, define

$$\xi_F(A) = \{x \in (0, 1] : \inf\{s : F(s) \geq x\} \in A\}.$$

If $A \in \mathcal{B}(\mathbb{R})$, then $\xi_F(A) \in \mathcal{B}((0, 1])$. Finally,

$$P_F(A) = \lambda(\xi_F(A)).$$

(Other HW 2 problems: Section 2.6, Q1-3, Q6, Q8-9, Q12, Q15-17, Q21, Q23)

Appendix: Proof of the Extension Theorem

As we mentioned before, the extension of P is from $\mathcal{S}$ to $\sigma(\mathcal{S})$. We prove this extension using two steps:

- Step 1: We extend P uniquely from $\mathcal{S}$ to $\mathcal{A}(\mathcal{S})$, the smallest algebra containing $\mathcal{S}$ (**First Extension Theorem**).
- Step 2: We extend P uniquely from $\mathcal{A}(\mathcal{S})$ to $\sigma(\mathcal{S})$ (**Second Extension Theorem**).

$$\text{Semi-algebra} \to \text{Algebra} \to \sigma\text{-algebra}.$$

Lemma 2.4.1 The algebra generated by a semi-algebra

Suppose $\mathcal{S}$ is a semi-algebra of subsets of Ω. Then $\mathcal{A}(\mathcal{S}) = \Lambda$ where

$$\Lambda = \{\cup_{i \in I} S_i : I \text{ a finite index set}, \{S_i : i \in I\} \text{ disjoint}, S_i \in \mathcal{S}\},$$

is the family of all unions of finite families of mutually disjoint subsets of Ω in $\mathcal{S}$.

Proof of Lemma 2.4.1

It is clear that $\mathcal{S} \subset \Lambda$. Now we check whether Λ is an algebra:

(i) $\Omega \in \mathcal{S}$, thus $\Omega \in \Lambda$.

(ii) Λ is closed under finite intersection.

(iii) To check closure under complementation, we see $(\cup_{i \in I} S_i)^c = \cap_{i \in I} S_i^c$. By the definition of $\mathcal{S}$, $S_i^c = \cup_{j \in J_i} C_{ij}$ for a finite index set J_i and mutually disjoint sets $\{C_{ij} : j \in J_i\}$. Thus

$$(\cup_{i \in I} S_i)^c = \cap_{i \in I} \cup_{j \in J_i} C_{ij} \in \Lambda.$$

Thus Λ is an algebra containing $\mathcal{S}$. Because $\mathcal{A}(\mathcal{S})$ is the algebra generated by $\mathcal{S}$, we conclude $\mathcal{A}(\mathcal{S}) \subset \Lambda$. On the other hand, because Λ is created by applying the finite union structure on $\mathcal{S}$, $\Lambda \subset \mathcal{A}(\mathcal{S})$. Finally, we have $\mathcal{A}(\mathcal{S}) = \Lambda$.

First Extension Theorem

From

$$\mathcal{S} = \{(x, y] : -\infty \leq x \leq y \leq \infty\}$$

to

$$\mathcal{A}(\mathcal{S}) = \{\cup_{i \in I} S_i : I \text{ a finite index set}, \{S_i : i \in I\} \text{ disjoint}, S_i \in \mathcal{S}\}.$$

Theorem 2.4.1 First Extension Theorem

Suppose $\mathcal{S}$ is a semialgrbra of subsets of Ω and $P : \mathcal{S} \mapsto [0, 1]$ is σ-additive on $\mathcal{S}$ and satisfies $P(\Omega) = 1$. There is a unique extension P' of P to $\mathcal{A}(\mathcal{S})$, defined by

$$P'(\cup_{i \in I} S_i) = \sum_{i \in I} P(S_i),$$

which is a probability measure on $\mathcal{A}(\mathcal{S})$; that is $P'(\Omega) = 1$ and $P' \geq 0$ is σ-additive on $\mathcal{A}(\mathcal{S})$.

Proof of First Extension Theorem

Obviously P' is an extension of P from $\mathcal{S}$ to $\mathcal{A}(\mathcal{S})$. Taking I to be a singleton index set, $P'(S) = P(S)$ for $S \in \mathcal{S}$.

1. Is P' defined unambiguously? Suppose $A = \cup_{i \in I} S_i = \cup_{j \in J} S_j$,

$$\begin{aligned}
\sum_{i \in I} P(S_i) &= \sum_{i \in I} P(S_i \cap A) = \sum_{i \in I} P(S_i \cap \{\cup_{j \in J} S_j\}) \\
&= \sum_{i \in I} P(\cup_{j \in J} \{S_i \cap S_j\}) = \sum_{i \in I} \sum_{j \in J} P(S_i \cap S_j) \\
&= \sum_{j \in J} \sum_{i \in I} P(S_j \cap S_i) = \sum_{j \in J} P(S_j).
\end{aligned}$$

2. Is P' σ-additive on $\mathcal{A}(\mathcal{S})$? Known P is σ-additive on $\mathcal{S}$.
Suppose for $i \geq 1$, $A_i = \sum_{i \in J_i} S_{ij} \in \mathcal{A}(\mathcal{S})$ for some $S_{ij} \in \mathcal{S}$ are mutually disjoint (thus $\{S_{ij} : i, j\}$ are mutually disjoint) and

$$A = \cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} \cup_{i \in J_i} S_{ij} \in \mathcal{A}(\mathcal{S}).$$

We need to show $P'(A) = \sum_{i=1}^{\infty} P'(A_i)$.
Because $A \in \mathcal{A}(\mathcal{S})$, thus A itself can be written as

$$A = \cup_{k \in K} S_k, P'(A) = \sum_{k \in K} P(S_k) \text{ for some } S_k \in \mathcal{S} \text{ and a finite } K.$$

Note $\mathcal{S}$ is a π-system, meaning $S_k \cap S_{ij} \in \mathcal{S}$. Because P is σ-additive on $\mathcal{S}$, $P(A \cap S_{ij}) = P(\cup_{k \in K}(S_k \cap S_{ij})) = \sum_{k \in K} P(S_k \cap S_{ij})$. Again, $P(\cup_{j \in J_i} S_{ij}) = \sum_{j \in J_i} P(S_{ij}) = P'(A_i)$. Thus

$$\sum_{i=1}^{\infty} P'(A_i) = \sum_{i=1}^{\infty} \sum_{j \in J_i} P(S_{ij}) = \sum_{i=1}^{\infty} \sum_{j \in J_i} \sum_{k \in K} P(S_k \cap S_{ij}) \underset{??}{=} \sum_{k \in K} P(S_k).$$

Proof of First Extension Theorem

Switching the order of those summations

$$\sum_{i=1}^{\infty} P'(A_i) = \sum_{k\in K} \sum_{i=1}^{\infty} \sum_{j\in J_i} P(S_k \cap S_{ij}) \underset{??}{=} \sum_{k\in K} P(S_k).$$

It suffices to show $\sum_{i=1}^{\infty} \sum_{j\in J_i} P(S_k \cap S_{ij}) = P(S_k)$.
Now play the trick $S_k = S_k \cap A = S_k \cap (\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty}(S_k \cap A_i) = \cup_{i=1}^{\infty} \cup_{j\in J_i} (S_k \cap S_{ij})$. We see that $S_k \in \mathcal{S}$ is a countable union of disjoint sets $S_k \cap S_{ij}$ in $\mathcal{S}$. Thus, by the σ-additivity of P on $\mathcal{S}$, we have $P(S_k) = \sum_{i=1}^{\infty} \sum_{j\in J_i} P(S_k \cap S_{ij})$ which completes the proof.

3. <u>Is P' unique?</u>
Suppose there two P_1' and P_2' σ-additive extensions of P from $\mathcal{S}$ to $\mathcal{A}(\mathcal{S})$, then for any

$$A = \cup_{i \in I} \mathcal{S}_i \in \mathcal{A}(\mathcal{S}),$$

by the σ-additivity, we have

$$P_1'(A) = \sum_{i \in I} P(\mathcal{S}_i) = P_2'(A).$$

Second Extension Theorem

Semi-algebra $\rightarrow$ Algebra $\rightarrow$ σ-algebra.

$$\mathcal{S} \rightarrow \quad \underbrace{\mathcal{A}(\mathcal{S}) \rightarrow \sigma\text{-algebra.}}_{\text{Second Extension Theorem}}$$

Theorem 2.4.2 Second Extension Theorem

A probability measure P defined on an algebra $\mathcal{A}$ of subsets has a unique extension to a probability measure on $\sigma(\mathcal{A})$.

The proof is (very long) broken into 3 parts:

- Part I: extend P to a σ-additive function Π on a class $\mathcal{G} \supset \mathcal{A}$.
- Part II: extend Π to a set function Π^* on a class $\mathcal{D} \supset \sigma(\mathcal{A})$.
- Part III: restrict Π^* to $\sigma(\mathcal{A})$ yielding the desired extension.

We begin by defining the class $\mathcal{G}$:

$$\begin{aligned}\mathcal{G} &= \left\{\cup_{j=1}^{\infty} A_j : A_j \in \mathcal{A}\right\} \\ &= \left\{\lim_{n\to\infty} B_n : B_n \in \mathcal{A}, B_n \subset B_{n+1} \forall n\right\}.\end{aligned}$$

That is $\mathcal{G}$ is the class of unions of countable collections of sets in $\mathcal{A}$, or equivalently, since $\mathcal{A}$ is an algebra, $\mathcal{G}$ is the class of non-decreasing limits of elements of $\mathcal{A}$ (think $B_n = \cup_{j=1}^{n} A_j$). Of course, $\mathcal{A} \subset \mathcal{G}$.
In Section 2.2: Every σ-algebra is a monotone class; An algebra is a σ-algebra iff it is a monotone class.
Now we define $\Pi : \mathcal{G} \mapsto [0,1]$ by: if $G = \lim_{n\to\infty} B_n \in \mathcal{G}$,

$$\Pi(G) = \lim_{n\to\infty} P(B_n).$$

Because P is σ-additive on $\mathcal{A}$, $\{P(B_n)\}$ is an increasing real sequence in $[0,1]$, thus $\Pi(G) = \lim_{n\to\infty} P(B_n)$ exists in $[0,1]$.

Proof of Second Extension Theorem: Part I

Furthermore, we need to check if $G = \lim_{n\to\infty} B_n = \lim_{n\to\infty} B'_n$ where both $B_n \uparrow$ and $B'_n \uparrow$ in $\mathcal{A}$, whether

$$\Pi(G) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P(B'_n).$$

For a fixed m, $B'_n \supset (B_m \cap B'_n)$ and $\{B_m \cap B'_n\}$ is an increasing sequence of sets, thus $\lim_{n\to\infty} P(B'_n) \geq \lim_{n\to\infty} P(B_m \cap B'_n)$. Because $B_m \cap B'_n \to B_m$ as $n \to \infty$, by the continuity (a consequence of being σ-additive) of P, we have $\lim_{n\to\infty} P(B_m \cap B'_n) = P(B_m)$. Therefore $\lim_{n\to\infty} P(B'_n) \geq \lim_{n\to\infty} P(B_m \cap B'_n) \geq P(B_m), \forall m$. Now take m to infinity, we have

$$\lim_{n\to\infty} P(B'_n) \geq \lim_{m\to\infty} P(B_m) = \lim_{n\to\infty} P(B_n).$$

Similarly, we have $\lim_{n\to\infty} P(B_n) \geq \lim_{n\to\infty} P(B'_n)$. Thus, $\lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P(B'_n)$.

Proof of Second Extension Theorem: Part I

Obviously, $\Pi(A) = P(A)$ for $A \in \mathcal{A}$ (take $B_n = A$). Therefore, we have created an extension Π of P from $\mathcal{A}$ to $\mathcal{G} \subset \mathcal{A}$.

Is Π a σ-additive extension of P? We answer this question by exploring the properties of Π and $\mathcal{G}$.

Property 1

Π is an extension of P from $\mathcal{A}$ to $\mathcal{G}$.

Proof of Property 1:

$\emptyset \in \mathcal{G}$, $\Pi(\emptyset) = 0$, $\Omega \in \mathcal{G}$, $\Pi(\Omega) = 1$, and for $G \in \mathcal{G}$, $0 \leq \Pi(G) \leq 1$. More generally, we have $\mathcal{A} \subset \mathcal{G}$ and $\Pi(A) = P(A)$ for $A \in \mathcal{A}$; i.e., $\Pi|_{\mathcal{A}} = P$.

Property 2: Π is additive on $\mathcal{G}$.

If $G_i \in \mathcal{G}$ for $i = 1, 2$ then $G_1 \cup G_2 \in \mathcal{G}$ and $G_1 \cap G_2 \in \mathcal{G}$, and

$$\Pi(G_1 \cup G_2) + \Pi(G_1 \cap G_2) = \Pi(G_1) + \Pi(G_2).$$

Proof of Property 2: By the definition of $\mathcal{G}$, we have $\mathcal{A} \ni B_{n1} \uparrow G_1$ and $\mathcal{A} \ni B_{n2} \uparrow G_2$. Since $\mathcal{A}$ is an algebra, we have $\mathcal{A} \ni B_{n1} \cup B_{n2} \uparrow G_1 \cup G_2 \in \mathcal{G}$ and $\mathcal{A} \ni B_{n1} \cap B_{n2} \uparrow G_1 \cap G_2 \in \mathcal{G}$. Further, because P is σ-additive on $\mathcal{A}$, we have

$$P(B_{n1} \cup B_{n2}) + P(B_{n1} \cap B_{n2}) = P(B_{n1}) + P(B_{n2})$$

holds for all n. Taking $n \to \infty$, we proved Property 2.

Property 3: Π is monotone on $\mathcal{G}$.

If $G_i \in \mathcal{G}$ for $i = 1, 2$ and $G_1 \subset G_2$, then

$$\Pi(G_1) \leq \Pi(G_2).$$

Proof of Property 3: (Similarly to Slide 22) By the definition of $\mathcal{G}$, we have $\mathcal{A} \ni B_{ni} \uparrow G_i$ for $i = 1, 2$ and $\cup_{n=1}^{\infty} B_{n1} = G_1 \subset G_2 = \cup_{n=1}^{\infty} B_{n2}$. For a fixed m, $B_{n2} \supset (B_{m1} \cap B_{n2})$ and $\{B_{m1} \cap B_{n2}\}$ is an increasing sequence of sets, thus $\lim_{n\to\infty} P(B_{n2}) \geq \lim_{n\to\infty} P(B_{m1} \cap B_{n2})$. Because $B_{m1} \cap B_{n2} \to B_{m1}$ as $n \to \infty$, by the continuity (a consequence of being σ-additive) of P, we have $\lim_{n\to\infty} P(B_{m1} \cap B_{n2}) = P(B_{m1})$. Therefore $\lim_{n\to\infty} P(B_{n2}) \geq \lim_{n\to\infty} P(B_{m1} \cap B_{n2}) \geq P(B_{m1})$ for every m. Now take m to infinity, we have

$$\Pi(G_2) = \lim_{n\to\infty} P(B_{n2}) \geq \lim_{m\to\infty} P(B_{m1}) = \Pi(G_1).$$

Proof of Second Extension Theorem: Part I

Property 4: $\mathcal{G}$ is closed under monotone limits and Π is monotonely continuous on $\mathcal{G}$.

If $G_n \in \mathcal{G}$ and $G_n \uparrow G$, then $G \in \mathcal{G}$ and $\Pi(G) = \lim_{n\to\infty} \Pi(G_n)$.

Proof of Property 4: For each n, we have $\mathcal{A} \ni B_{m,n} \uparrow G_n$.
Now define $D_m = \cup_{n=1}^{m} B_{m,n}$.
Since $\mathcal{A}$ is closed under finite unions, $D_m \in \mathcal{A}$.
We show $D_m \uparrow G$.
It is easy to see that $\{D_m\}$ is monotone: $D_m = \cup_{n=1}^{m} B_{m,n} \subset \cup_{n=1}^{m} B_{m+1,n} \subset \cup_{n=1}^{m+1} B_{m+1,n} = D_{m+1}$.
If $n \leq m$, we also have $B_{m,n} \subset D_m = \cup_{j=1}^{m} B_{m,j} \subset \cup_{j=1}^{m} G_j = G_m$.
Taking limits on m, we have for any $n \geq 1$,
$G_n = \lim_{m\to\infty} B_{m,n} \subset \lim_{m\to\infty} D_m \subset \lim_{m\to\infty} G_m = G$.
Now taking limits on n yields $G = \lim_{n\to\infty} G_n \subset \lim_{m\to\infty} D_m \subset G$.
Thus $D_m \uparrow G$ and proves $G \in \mathcal{G}$.
Furthermore, by the definition of Π, we have $\Pi(G) = \lim_{n\to\infty} \Pi(D_m)$.

Proof of Second Extension Theorem: Part I

It remains to show $\Pi(G_n) \uparrow \Pi(G)$.
By the monotonicity of Π on $\mathcal{G}$,

$$\Pi(B_{m,n}) \leq \Pi(D_m) \leq \Pi(G_m).$$

Let $m \to \infty$, $B_{m,n} \uparrow G_n$,

$$\Pi(G_n) = \lim_{m\to\infty} \Pi(B_{m,n}) \leq \lim_{m\to\infty} \Pi(D_m) \leq \lim_{m\to\infty} \Pi(G_m), \forall n$$

Let $n \to \infty$,

$$\lim_{n\to\infty} \Pi(G_n) \leq \lim_{m\to\infty} \Pi(D_m) \leq \lim_{m\to\infty} \Pi(G_m).$$

Therefore

$$\lim_{n\to\infty} \Pi(G_n) = \lim_{m\to\infty} \Pi(D_m) = \Pi(G).$$

Proof of Second Extension Theorem: Part I

Property 5: Π is σ-additive on $\mathcal{G}$.

If $\{A_i : i \geq 1\}$ is a disjoint sequence of sets in $\mathcal{G}$, by Property 2, we have $G_n = \cup_{i=1}^n A_i \in \mathcal{G}$, by Property 4, we have $\cup_{i=1}^\infty A_i = \lim_{n\to\infty} G_n \in \mathcal{G}$ and

$$\begin{aligned}\Pi(\cup_{i=1}^\infty A_i) &= \Pi(\lim_{n\to\infty} G_n) = \lim_{n\to\infty} \Pi(G_n) \\ &= \lim_{n\to\infty} \sum_{i=1}^n \Pi(A_i) = \sum_{i=1}^\infty \Pi(A_i).\end{aligned}$$

Proof of Second Extension Theorem: Part II

Part I has extended $\mathcal{P}$ to a σ-additive Π from the algebra $\mathcal{A}$ to the monotone class

$$\mathcal{G} = \{\lim_{n\to\infty} B_n : B_n \in \mathcal{A}, B_n \subset B_{n+1}, \forall n\}.$$

Part II extends Π to a set function Π^* on the power set $\mathcal{P}(\Omega)$ (the largest σ-algebra on Ω) and finally show that the restriction of Π^* to a certain subclass $\mathcal{D}$ of $\mathcal{P}(\Omega)$ can yield the desired extension of P.

We define $\Pi^* : \mathcal{P}(\Omega) \mapsto [0, 1]$ by

$$\forall A \in \mathcal{P}(\Omega) : \Pi^*(A) = \inf\{\Pi(G) : A \subset G \in \mathcal{G}\},$$

so $\Pi^*(A)$ is the least upper bound of values of Π on sets $G \in \mathcal{G}$ containing A.

Proof of Second Extension Theorem: Part II

We now consider properties of Π^*:

Property 1.

We have on $\mathcal{G}$:

$$\Pi^*|_{\mathcal{G}} = \Pi$$

and $0 \leq \Pi^*(A) \leq 1$ for any $A \in \mathcal{P}(\Omega)$.

Proof of Property 1:

For $A \in \mathcal{G}$, Then for any $\mathcal{G} \ni G \supset A$, we have $\Pi(G) \geq \Pi(A)$.
Known $A \subset A$, thus $\Pi^*(A) = \inf\{\Pi(G) : A \subset G \in \mathcal{G}\} = \Pi(A)$.

In particular, we have $\Pi^*(\Omega) = \Pi(\Omega) = 1$ and $\Pi^*(\emptyset) = \Pi(\emptyset) = 0$.

Proof of Second Extension Theorem: Part II

Property 2.

For $A_1, A_2 \in \mathcal{P}(\Omega)$,

$$\Pi^*(A_1 \cup A_2) + \Pi^*(A_1 \cap A_2) \leq \Pi^*(A_1) + \Pi^*(A_2)$$

and consequently $1 = \Pi^*(\Omega) \leq \Pi^*(A) + \Pi^*(A^c)$.

Proof of Property 2:

$\forall \epsilon > 0$, find $G_i \in \mathcal{G}$ such that $G_i \supset A_i$ and

$$\Pi^*(A_i) + \epsilon/2 \geq \Pi(G_i).$$

Thus

$$\begin{aligned}\Pi^*(A_1) + \Pi^*(A_2) + \epsilon \geq \Pi(G_1) + \Pi(G_2) &= \Pi(G_1 \cup G_2) + \Pi(G_1 \cap G_2) \\ &\geq \Pi^*(A_1 \cup A_2) + \Pi^*(A_1 \cap A_2).\end{aligned}$$

Proof of Second Extension Theorem: Part II

Property 3.

Π^* is monotone on $\mathcal{P}(\Omega)$.

Proof of Property 3:

This follows the fact that Π is monotone on $\mathcal{G}$. For $A_1 \subset A_2 \in \mathcal{P}(\Omega)$,

$$\begin{aligned} \Pi^*(A_1) = \ & \inf\{\Pi(G_1) : A_1 \subset G_1 \in \mathcal{G}\} \\ \leq \ & \inf\{\Pi(G_2) : A_1 \subset A_2 \subset G_2 \in \mathcal{G}\} = \Pi^*(A_2). \end{aligned}$$

Proof of Second Extension Theorem: Part II

Property 4.

Π^* is sequentially monotone continuous on $\mathcal{P}(\Omega)$ in the sense that if $A_n \uparrow A$, then $\Pi^*(A_n) \uparrow \Pi^*(A)$.

Proof of Property 4:

Fix $\epsilon > 0$, for each $n \geq 1$, find $G_n \in \mathcal{G}$ such that $A_n \subset G_n$ and $\Pi^*(A_n) + \epsilon/2^n \geq \Pi(G_n)$. Define $G_n' = \cup_{m=1}^n G_m$. Since $\mathcal{G}$ is closed under finite unions $G_n' \in \mathcal{G}$ and G_n' is obviously non-decreasing. We claim for all $n \geq 1$,

$$\Pi^*(A_n) + \epsilon \sum_{i=1}^{n} 2^{-i} \geq \Pi(G_n').$$

Proof of this claim is by induction. It certainly holds for $n = 1$. Suppose it holds for n, we prove it also holds for $n + 1$.

Proof of Second Extension Theorem: Part II

Proof of Property 4:

We have $A_n \subset G_n \subset G'_n$ and $A_n \subset A_{n+1} \subset G_{n+1}$, and consequently $A_n \subset G'_n$ and $A_n \subset G_{n+1}$. So $A_n \subset G'_n \cap G_{n+1} \in \mathcal{G}$. Thus

$$\begin{aligned}
\Pi(G'_{n+1}) = \Pi(G'_n \cup G_{n+1}) &= \Pi(G'_n) + \Pi(G_{n+1}) - \Pi(G'_n \cap G_{n+1}) \\
&\leq \Pi^*(A_n) + \epsilon \sum_{i=1}^{n} 2^{-i} + \Pi^*(A_{n+1}) + \epsilon 2^{-n-1} - \Pi^*(A_n) \\
&= \Pi^*(A_{n+1}) + \epsilon \sum_{i=1}^{n+1} 2^{-i}.
\end{aligned}$$

This finishes the proof of the claim. Now in the claim, we let $n \to \infty$, by monotonicity of $\{\Pi^*(A_n)\}$ and $\{\Pi(G'_n)\}$ (limits exist),

$$\lim_{n\to\infty} \Pi^*(A_n) + \epsilon \geq \lim_{n\to\infty} \Pi(G'_n) = \Pi(\cup_{j=1}^{\infty} G'_j).$$

Proof of Property 4:

Since

$$A = \lim_{n\to\infty} A_n \subset \cup_{j=1}^{\infty} G'_j \in \mathcal{G}.$$

We conclude (let $\epsilon \to 0$),

$$\lim_{n\to\infty} \Pi^*(A_n) \geq \Pi(\cup_{j=1}^{\infty} G'_j) \geq \Pi^*(A).$$

On the other hand, by Property 3, we have $\Pi^*(A_n)$ and $\Pi^*(A_n) \leq \Pi^*(A)$. Thus $\lim_{n\to\infty} \Pi^*(A_n) \leq \Pi^*(A)$ which proves

$$\lim_{n\to\infty} \Pi^*(A_n) = \Pi^*(A).$$

Proof of Second Extension Theorem: Part III

Did we prove Π^* is σ-additive on $\mathcal{P}(\Omega)$? No, because of the $\leq$ sign in Property 2 on Slide 31, and also because $1 = \Pi^*(\Omega) \leq \Pi^*(A) + \Pi^*(A^c)$.

So far, **Part I** has extended $\mathcal{P}$ to a σ-additive Π from the algebra $\mathcal{A}$ to the monotone class $\mathcal{G} = \{\lim_{n\to\infty} B_n : B_n \in \mathcal{A}, B_n \subset B_{n+1}, \forall n\}$.
Part II extends Π to a set function Π^* (which might not be σ-additive) on the power set $\mathcal{P}(\Omega)$.
Part III: We now retract Π^* to a certain subclass $\mathcal{D}$ of $\mathcal{P}(\Omega)$ and show that $\Pi^*|_{\mathcal{D}}$ is the desired extension of P from $\mathcal{A}$ to $\sigma(\mathcal{A}) \subset \mathcal{D}$, where

$$\mathcal{D} = \{D \in \mathcal{P}(\Omega) : \Pi^*(D) + \Pi^*(D^c) = 1\}.$$

Obviously, $\mathcal{A} \subset \mathcal{D}$, because if $A \in \mathcal{A}$, then $\Pi^*(A) = \Pi(A) = 1 - \Pi(A^c)$.

Proof of Second Extension Theorem: Part III

$$\mathcal{D} = \{D \in \mathcal{P}(\Omega) : \Pi^*(D) + \Pi^*(D^c) = 1\}.$$

Lemma 2.4.3

The class $\mathcal{D}$ has the following properties:

1. $\mathcal{D}$ is a σ-field.
2. $\Pi^*|_{\mathcal{D}}$ is a probability measure on $(\Omega, \mathcal{D})$.

If Lemma 2.4.3 is true, we know that $\mathcal{A} \subset \mathcal{D}$ and thus $\sigma(\mathcal{A}) \subset \mathcal{D}$. The restriction $\Pi^*|_{\sigma(\mathcal{A})}$ is the desired extension of P on $\mathcal{A}$ to a probability measure on $\sigma(\mathcal{A})$.

Proof of Second Extension Theorem: Part III

Proof of Lemma 2.4.3

We first show that $\mathcal{D}$ is an algebra. Obviously Ω and $\emptyset$ are in $\mathcal{D}$, and obviously $\mathcal{D}$ is closed under complementation. Next we show $\mathcal{D}$ is closed under finite unions and finite intersections. For $D_1, D_2 \in \mathcal{D}$, from Property 2 of Π^* on $\mathcal{P}(\Omega)$:

$$\Pi^*(D_1 \cup D_2) + \Pi^*(D_1 \cap D_2) \leq \Pi^*(D_1) + \Pi^*(D_2)$$
$$\Pi^*(D_1^c \cup D_2^c) + \Pi^*(D_1^c \cap D_2^c) \leq \Pi^*(D_1^c) + \Pi^*(D_2^c).$$

Adding them together yields

$$2 \leq \underbrace{\Pi^*(D_1 \cup D_2) + \Pi^*(D_1^c \cap D_2^c)}_{=1} + \underbrace{\Pi^*(D_1 \cap D_2) + \Pi^*(D_1^c \cup D_2^c)}_{=1} \leq 2.$$

This completes the proof of $\mathcal{D}$ being an algebra. In addition, we must have $\Pi^*(D_1 \cup D_2) + \Pi^*(D_1 \cap D_2) = \Pi^*(D_1) + \Pi^*(D_2)$ for any D_1 and D_2 in $\mathcal{D}$. Thus Π^* is finitely additive on $\mathcal{D}$.

Proof of Lemma 2.4.3

We now show $\mathcal{D}$ is a σ-algebra. Recall that in Section 2.2: An algebra is a σ-algebra iff it is a monotone class. It suffices to show $\mathcal{D}$ is a monotone class. Because of the closeness under complementation, it is enough to focus on monotone increasing; i.e., if $D_n \in \mathcal{D}$, $D_n \uparrow D$ implies $D \in \mathcal{D}$.

We know $\mathcal{D} \in \mathcal{P}(\Omega)$. By Properties 3 and 2 of Π^* on $\mathcal{P}(\Omega)$, $\lim_{n\to\infty} \Pi^*(D_n) = \Pi^*(D)$, and $1 \leq \Pi^*(D) + \Pi^*(D^c)$.
$\Pi^*(D^c) = \Pi^*((\cup_{n=1}^{\infty} D_n)^c) = \Pi^*(\cap_{n=1}^{\infty} D_n^c) \leq \Pi^*(D_m^c)$ for any $m \geq 1$. Thus $1 \leq \Pi^*(D) + \Pi^*(D^c) \leq \Pi^*(D) + \Pi^*(D_m^c) = \lim_{n\to\infty} \Pi^*(D_n) + \Pi^*(D_m^c)$. We know $\{\Pi^*(D_m^c)\}$ is a bounded monotone non-increasing sequence. Its limit exists. Thus taking m to infinity,
$1 \leq \Pi^*(D) + \Pi^*(D^c) \leq \lim_{n\to\infty} \Pi^*(D_n) + \lim_{m\to\infty} \Pi^*(D_m^c) = \lim_{n\to\infty}\{\Pi^*(D_n) + \Pi^*(D_n^c)\} = 1$.

Proof of Lemma 2.4.3

Finally, we show $\Pi^*|_{\mathcal{D}}$ is σ-additive. Let $\{D_n\}$ be a sequence of disjoint sets in $\mathcal{D}$. Because Π^* is continuous with respect to non-decreasing sequences (Property 4), we have

$$\Pi^*(\cup_{n=1}^{\infty} D_n) = \Pi^*(\lim_{n\to\infty} \cup_{i=1}^{n} D_i) = \lim_{n\to\infty} \Pi^*(\cup_{i=1}^{n} D_i).$$

Because Π^* is finitely additive on $\mathcal{D}$, $\Pi^*(\cup_{i=1}^{n} D_i) = \sum_{i=1}^{n} \Pi^*(D_i)$, we have

$$\Pi^*(\cup_{n=1}^{\infty} D_n) = \lim_{n\to\infty} \Pi^*(\cup_{i=1}^{n} D_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \Pi^*(D_i) = \sum_{i=1}^{\infty} \Pi^*(D_i).$$

We have finished the proof of Lemma 2.4.3.

So far, we have shown that $\Pi^*|_{\sigma(\mathcal{A})}$ is the desired extension of P on $\mathcal{A}$ to a probability measure on $\sigma(\mathcal{A})$.

Proof of Second Extension Theorem: Part III

Did we finish the proof of the Second Extension Theorem?
No, we need to show uniqueness.

But it is trivial because of Corollary 2.2.1: If P_1, P_2 are two probability measures on $(\Omega, \mathcal{B})$ and if $\mathcal{P}$ is a π-system such that $\forall A \in \mathcal{P}$, $P_1(A) = P_2(A)$, then $\forall B \in \sigma(\mathcal{P})$, $P_1(B) = P_2(B)$.

If we have two distinct extensions from $\mathcal{A}(\mathcal{S})$ to $\sigma(\mathcal{A}(\mathcal{S}))$, then they must be the same on $\sigma(\mathcal{A}(\mathcal{S}))$ because $\mathcal{A}(\mathcal{S})$ is a π-system.

Summary

First Extension Theorem: uniquely,

$$\overbrace{\underbrace{P \text{ on } \mathcal{S}}_{Semi-algebra}}^{\sigma-additive} \rightarrow \overbrace{\underbrace{P' \text{ on } \mathcal{A}(\mathcal{S})}_{algebra}}^{\sigma-additive}.$$

Second Extension Theorem: uniquely,

$$\overbrace{\underbrace{P' \text{ on } \mathcal{A}(\mathcal{S})}_{algebra}}^{\sigma-additive} \rightarrow \overbrace{\underbrace{\Pi \text{ on } \mathcal{G}}_{monotone\ \ class}}^{\sigma-additive} \rightarrow \overbrace{\underbrace{\Pi^{*} \text{ on } \mathcal{P}(\Omega)}_{largest\ \ \sigma-algebra}}^{not\ \ \sigma-additive}$$

and

$$\overbrace{\underbrace{\Pi^{*} \text{ on } \mathcal{P}(\Omega)}_{largest\ \ \sigma-algebra}}^{not\ \ \sigma-additive} \xrightarrow{restriction} \overbrace{\underbrace{\Pi^{*} \text{ on } \mathcal{D}}_{smaller\ \ \sigma-algebra}}^{\sigma-additive} \xrightarrow{restriction} \overbrace{\underbrace{\Pi^{*} \text{ on } \sigma(\mathcal{A}(\mathcal{S}))}_{GOAL}}^{\sigma-additive}.$$