STAT 810 Probability Theory I

Chapter 4: Independence

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Introduction

Independence is a basic property of events and random variables in a probability model. Its intuitive appeal stems from the easily envisioned property that the occurrence or non-occurrence of an event has no effect on our estimate of the probability that an independent event will or will not occur.

Despite the intuitive appeal, it is important to recognize that independence is a technical concept with a technical definition which must be checked with respect to a specific probability model. There are examples of dependent events which intuition insists must be independent, and examples of events which intuition insists cannot be independent but still satisfy the definition.

One really must check the technical definition to be sure.

4.1 Basic Definitions

Definition 4.1.1 Independence for two events

Suppose $(\Omega, \mathcal{B}, P)$ is a fixed probability space. Events $A, B \in \mathcal{B}$ are independent if

$$P(A \cap B) = P(A)P(B).$$

Definition 4.1.2 Independence of a finite number of events

The events $A_1, \ldots, A_n$ $(n \geq 2)$ are independent if

$$P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i), \quad \text{fro all finite } I \subset \{1, \ldots, n\}.$$

Definition 4.1.3 Independent classes

Let $\mathcal{C}_i \subset \mathcal{B}$, $i = 1, \ldots, n$. The classes $\mathcal{C}_i$ are independent, if for any choice $A_1, \ldots, A_n$ with $A_i \in \mathcal{C}_i$, $i = 1, \ldots, n$, we have the events $A_1, \ldots, A_n$ independent events.

Theorem 4.1.1 (Basic Criterion)

If for each $i = 1, \ldots, n$, $\mathcal{C}_i$ is a non-empty class of events satifying

1. $\mathcal{C}_i$ is a π-system
2. $\mathcal{C}_i$, $i = 1, \ldots, n$, are independent, then

$$\sigma(\mathcal{C}_1), \ldots, \sigma(\mathcal{C}_n) \quad \text{are independent.}$$

Proof of this uses a Dynkin system (λ-system). Recall that $\mathcal{L}$ is a Dynkin system if (1) $\Omega \in \mathcal{L}$; (2) $A \in \mathcal{L}$ imples $A^c \in \mathcal{L}$; (3) if $B_n \in \mathcal{L}$ are disjoint, $\cup_n B_n \in \mathcal{L}$. (A λ-system is a σ-algebra iff it is a π system).

Dynkin's Theorem 2.2.2 If $\mathcal{P}$ is a π-system, $\mathcal{L}$ is a λ-system, and $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.

Proof of Theorem 4.1.1: We only prove it for $n = 2$, by induction it easily gets to $n > 2$. Fix $A_2 \in \mathcal{C}_2$. Let

$$\mathcal{L} = \{A \in \mathcal{B} : P(A \cap A_2) = P(A)P(A_2)\}.$$

We claim $\mathcal{L}$ is a λ-system:

(1) $\Omega \in \mathcal{L}$ is obvious.

(2) if $A \in \mathcal{L}$, we have $P(A^c)P(A_2) = P(A_2)(1 - P(A)) = P(A_2) - P(A_2 \cap A) = P(A_2 \cap A^c)$; i.e., $A^c \in \mathcal{L}$.

(3) if $B_n \in \mathcal{L}$ are disjoint, $P((\cup_n B_n) \cap A_2) = P(\cup_n (B_n \cap A_2)) = \sum_n P(B_n \cap A_2) = \sum_n P(B_n)P(A_2) = P(\cup_n B_n)P(A_2)$.

Also $\mathcal{C}_1 \subset L$. By Dynkin's Theorem, $\sigma(\mathcal{C}_1) \subset \mathcal{L}$. Thus $\sigma(\mathcal{C}_1)$ and $\mathcal{C}_2$ are independent. Then fix $A_1 \in \sigma(\mathcal{C}_1)$ and define $\mathcal{L} = \{A \in \mathcal{B} : P(A \cap A_1) = P(A)P(A_1)\}$. It is easy to show $\sigma(\mathcal{C}_2)$ and $\sigma(\mathcal{C}_1)$ are independent.

Definition 4.1.4 Arbitrary number of independent classes

Let T be an arbitray index set. The classes $\mathcal{C}_t$, $t \in T$ are independent families if for any finite I, $I \subset T$, $\{C_t : t \in I\}$ are independent.

Corollary 4.1.1

If $\{\mathcal{C}_t : t \in T\}$ are non-empty π-systems that are independent. Then $\{\sigma(\mathcal{C}_i) : t \in T\}$ are independent.

Definition 4.2.1 Independent Random Variables

$\{X_t : t \in T\}$ is an independent family of random variables if $\{\sigma(X_t) : t \in T\}$ are independent σ-algebras.

Random variables are independent if their induced σ-algebras are independent.

We now give a criterion for independence of random variables in terms of distribution functions. For a family of random variables $\{X_t : t \in T\}$ indexed by a set T, the **finite dimensional distribution functions** are the family of multivariate distribution functions

$$F_J(x_t, t \in J) = P[X_t \leq x_t, t \in J]$$

for all finite subsets $J \subset T$.

Theorem 4.2.1 Factorization Criterion

A family of random variables $\{X_t : t \in T\}$ indexed by a set T, is independent iff for all finite $J \subset T$

$$F_J(x_t, t \in J) = \prod_{t \in J} P[X_t \leq x_t], \quad \forall x_t \in \mathbb{R}. \tag{1}$$

Proof: By Definition 4.1.1, it suffices to show that for a finite index set J that $\{X_t : t \in J\}$ are independent iff (1) holds. Define $\mathcal{C}_t = \{[X_t \leq x] : x \in \mathbb{R}\}$. Then $\sigma(X_t) = \sigma(\mathcal{C}_t)$ and $\mathcal{C}_t$ is a π-system. (1) says $\{\mathcal{C}_t : t \in J\}$ is an independent family and therefore by the Basic Criterion 4.1.1, $\{\sigma(\mathcal{C}_t) = \sigma(X_t) : t \in J\}$ are independent.

Corollary 4.2.1

The finite collection of random variables $X_1, \ldots, X_k$ is independent iff

$$P[X_1 \leq x_1, \ldots, X_k \leq x_k] = \prod_{i=1}^{k} P[X_i \leq x_i],$$

for all $x_i \in \mathbb{R}$, $i = 1, \ldots, k$.

Corollary 4.2.2

The discrete random variables $X_1, \ldots, X_k$ with countable range $\mathcal{R}$ are independent iff

$$P[X_1 = x_1, \ldots, X_k = x_k] = \prod_{i=1}^{k} P[X_i = x_i],$$

for all $x_i \in \mathcal{R}$, $i = 1, \ldots, k$.

Notation: $\perp$: $X \perp Y$, $A \perp B$, $A \perp \mathcal{C}$.

Let $\{X_n : n \geq 1\}$ be iid with common continuous distribution function $F(x)$. The continuity of F implies

$$P[X_i = X_j] = 0,$$

so that if we define

$$[Ties] = \cup_{i \neq j}[X_i = X_j].$$

then

$$P[Ties] = 0. Why?$$

Call X_n a **record** of the sequence if

$$X_n \geq \bigvee_{i=1}^{n-1} X_i,$$

and define

$$A_n = [X_n \text{ is a record}].$$

A result due to Renyi says that the events $\{A_j : j \geq 1\}$ are independent and

$$P(A_j) = j^{-1}, \quad j \geq 2.$$

This is a special case of a result about **relative ranks**.

Let R_n be the relative rank of X_n among $X_1, \dots, X_n$ where $R_n = \sum_{j=1}^{n} I_{[X_j \geq X_n]}$. So $R_n = 1$ iff X_n is a record, $R_n = 2$ iff X_n is the second largest of $X_1, \dots, X_n$, and so on.

Theorem 4.3.1 Renyi Theorem

Assume $\{X_n : n \geq 1\}$ are iid with common continuous distribution function $F(x)$.

(a) $\{R_n : n \geq 1\}$ are independent and

$$P[R_n = k] = \frac{1}{n}, \text{ for } k = 1, \dots, n.$$

(b) $\{A_n : n \geq 1\}$ are independent and

$$P(A_n) = \frac{1}{n}.$$

Proof of Renyi Theorem: (b) comes from (a) since $A_n = [R_n = 1]$. It suffices to show (a). There are $n!$ orderings of $X_1, \ldots, X_n$. Because X_i's are iid, all possible orderings have the same probability $1/(n!)$. Each realization of $R_1, \ldots, R_n$ uniquely determines an ordering. Thus $P[R_1 = r_1, \ldots, R_n = r_n] = 1/(n!)$, for $r_i \in \{1, \ldots, i\}$, $i = 1, \ldots, n$. Then

$$P[R_n = r_n] = \sum_{r_1, \ldots, r_{n-1}} \frac{1}{n!}.$$

Since r_i ranges over i values, the number of terms in the above sum is $(n-1)!$. Thus $P[R_n = r_n] = 1/n$ for $n = 1, 2, \ldots$. Therefore $P[R_1 = r_1, \ldots, R_n = r_n] = P[R_1 = r_1] \cdots P[R_n = r_n]$.

4.3.2 Dyadic Expansions of Uniform Random Numbers

Consider

$$(\Omega, \mathcal{B}, P) = ((0,1], \mathcal{B}((0,1]), \lambda),$$

where λ is Lebesgue measure. We write $\omega \in (0,1]$ using its dyadic expansion

$$\omega = \sum_{n=1}^{\infty} \frac{d_n(\omega)}{2^n} = .d_1(\omega)d_2(\omega)d_3(\omega)\cdots,$$

where each $d_n(\omega)$ is either 0 or 1. This expansion is not unique; e.g., $0.5 = \frac{1}{2} = 0.1 = 0.0111\cdots$. If it happens, we agree to use the non-terminating one; i.e., $0.5 = 0.01111\cdots$.

Fact 1. Each d_n is a binary random variable.

Fact 2. $P[d_n = 1] = 0.5 = P[d_n = 0]$

Fact 3. The sequence $\{d_n : n \geq 1\}$ is iid.

For Fact 1, it suffices to check $[d_n = 1] \in \mathcal{B}((0,1])$. When $n = 1$, $[d_n = 1] = (0.100\cdots, 0.111\cdots] = (\frac{1}{2}, 1] \in \mathcal{B}((0,1])$. For $n \geq 2$, $[d_n = 1] = \cup_{(u_1,\ldots,u_{n-1} \in \{0,1\}^{n-1}}(0.u_1\cdots u_{n-1}1000\cdots, 0.u_1\cdots u_{n-1}1111\cdots$ which is a disjoint union of 2^{n-1} intervals; e.g., $[d_2 = 1] = (\frac{1}{4}, \frac{1}{2}] \cup (\frac{3}{4}, 1]$. Thus $[d_n = 1] \in \mathcal{B}((0,1])$.

For Fact 2, we first see that $P((0.u_1\cdots u_{n-1}1000\cdots, 0.u_1\cdots u_{n-1}1111\cdots$ $\frac{1}{2^n}$. Thus $P[d_n = 1] = 2^{n-1}\frac{1}{2^n} = 1/2$.

For Fact 3, only independence is left. For $(u_1, \ldots, u_n) \in \{0,1\}^n$, we have

$$\begin{aligned} P(\cap_{i=1}^n [d_i = u_i]) &= P((0.u_1\cdots u_n 0000\cdots, 0.u_1\cdots u_n 1111\cdots]) \\ &= \frac{1}{2^n} = \prod_{i=1}^n P[d_i = u_i]. \end{aligned}$$

Lemma 4.4.1 (Grouping Lemma)

Let $\{\mathcal{B}_t : t \in T\}$ be an independent family of σ-algebras. Let S be an index set and suppose for $s \in S$ that $T_s \subset T$ and $\{T_s : s \in S\}$ is pairwise disjoint. Now define

$$\mathcal{B}_{T_s} = \bigvee_{t \in T_s} \mathcal{B}_t = \sigma(\mathcal{B}_t : t \in T_s)$$

Then

$$\{\mathcal{B}_{T_s} : s \in \mathcal{S}\}$$

is an independent family of σ-algebras.

Examples: (a) $\{X_n : n \geq 1\}$ are independent, then $\sigma(X_j : j \leq n)$ and $\sigma(X_j : j > n)$ are independent, so are $\sum_{i=1}^{n} X_i$ and $\sum_{i=n+1}^{n+k} X_i$, $\max_{i=1}^{n} X_i$ and $\max_{i=n+1}^{n+k} X_i$. (b) $\{A_n\}$ are independent events. $\cup_{n=1}^{N} A_n$ and $\cup_{j=N+1}^{\infty} A_j$ are independent.

Proof of Lemma 4.4.1: We only need focus on the case where S is finite. Define

$$\mathcal{C}_{T_s} = \{\cap_{\alpha \in K} B_\alpha : B_\alpha \in \mathcal{B}_\alpha, K \subset T_s, K \text{ is finite.}\}$$

Then $\mathcal{C}_{T_s}$ is a π-system for each s, and $\{\mathcal{C}_{T_s} : s \in S\}$ are independent classes.

By the Basic Criterion 4.1.1, it suffices to show $\sigma(\mathcal{C}_{T_s}) = \mathcal{B}_{T_s}$. It is obvious that $\sigma(\mathcal{C}_{T_s}) \subset \mathcal{B}_{T_s}$. Also, $\mathcal{B}_\alpha \subset \mathcal{C}_{T_s}$, for each $\alpha \in T_s$ (when $K = \{\alpha\}$). Hence $\cup_{\alpha \in T_s} \mathcal{B}_\alpha \subset \sigma(\mathcal{C}_{T_s})$ which completes the proofs.

4.5.1 Borel-Cantelli Lemma

Proposition 4.5.1 (Borel-Cantelli Lemma)

Let $\{A_n\}$ be any events. If

$$\sum_n P(A_n) < \infty,$$

the

$$P([A_n \; i.o.]) = P(\limsup_{n\to\infty} A_n) = 0.$$

Proof:

$$\begin{aligned}
\sum_n P(A_n) < \infty \implies 0 &= \limsup_{n\to\infty} \sum_{j=n}^{\infty} P(A_j) \\
&\geq \lim_{n\to\infty} P(\cup_{j\geq n} A_j) \\
&= P(\lim_{n\to\infty} \cup_{j\geq n} A_j) \\
&= P(\limsup_{n\to\infty} A_n) = P([A_n \; i.o.]).
\end{aligned}$$

Example 4.5.1

Suppose $\{X_n : n \geq 1\}$ are Bernoulli random variables (could be dependent) with $P[X_n = 1] = p_n$ (could vary with n). Then

$$P[\lim_{n\to\infty} X_n = 0] = 1 \text{ if } \sum_n p_n < \infty.$$

Proof: By applying the Borel-Cantelli lemma, we have $\sum_n P[X_n = 1] < \infty$ imply $0 = P([X_n = 1],\ i.o.) = P(\limsup_{n\to\infty}[X_n = 1])$. Taking complements, $1 = P(\liminf_{n\to\infty}[X_n = 0]) = P(\cup_{n\geq 1} \cap_{k\geq n} \{\omega : X_k(\omega) = 0\})$. If $\omega \in \cup_{n\geq 1} \cap_{k\geq n} \{\omega : X_k(\omega) = 0\}$, then there exists an $n \geq 1$, for all $k \geq n$, $X_k(\omega) = 0$, implying $\lim_{n\to\infty} X_n(\omega) = 0$. Thus $\liminf_{n\to\infty}[X_n = 0] \subset [\lim_{n\to\infty} X_n = 0]$. Therefore, $P([\lim_{n\to\infty} X_n = 0]) = 1$.

Proposition 4.5.2 (Borel Zero-One Law)

If $\{A_n\}$ is a sequence of independent events, then

$$P([A_n\ i.o.]) = \begin{cases} 0, & \text{iff } \sum_n P(A_n) < \infty, \\ 1, & \text{iff } \sum_n P(A_n) = \infty. \end{cases}$$

Proof: $\sum_n P(A_n) < \infty \implies P([A_n\ i.o.]) = 0$. Conversely, suppose $\sum_n P(A_n) = \infty$. We have $P([A_n\ i.o.]) =$

$$P(\limsup_{n\to\infty} A_n) = 1 - P(\liminf_{n\to\infty} A_n) = 1 - \lim_{n\to\infty} P(\cap_{k\geq n} A_k^c)$$

$$= 1 - \lim_{n\to\infty} \lim_{m\to\infty} P(\cap_{k=n}^m A_k^c) = 1 - \lim_{n\to\infty} \lim_{m\to\infty} \prod_{k=n}^{m} \{1 - P(A_k)\}.$$

4.5.2 Borel Zero-One Law

Proof (continued): It suffices to show $\lim_{n\to\infty}\lim_{m\to\infty}\prod_{k=n}^{m}\{1-P(A_k)\}=0$. Known that $1-x\le e^{-x}$ for $0<x<1$ and $\sum_n P(A_n)=\infty$,

$$\lim_{m\to\infty}\prod_{k=n}^{m}\{1-P(A_k)\}\le \lim_{m\to\infty}\prod_{k=n}^{m}e^{-P(A_k)}=\lim_{m\to\infty}e^{-\sum_{k=n}^{m}P(A_k)}$$
$$=e^{-\sum_{k=n}^{\infty}P(A_k)}=e^{-\infty}=0.$$

Example 4.5.1 (continued)

Suppose $\{X_n : n\ge 1\}$ are Bernoulli random variables (could be dependent) with $P[X_n=1]=p_n$ (could vary with n). We assert that

$$P[\lim_{n\to\infty}X_n=0]=1 \text{ iff } \sum_n p_n<\infty$$

.

Example 4.5.2 (Behavior of exponential random variables)

Suppose $\{E_n : n \geq 1\}$ are iid unit exponential random variables; that is $P[E_n > x] = e^{-x}$ for $x > 0$. Then

$$P[\limsup_{n\to\infty} E_n / \log n = 1] = 1.$$

Proof: For any $\omega \in \Omega$ such that

$$1 = \limsup_{n\to\infty} \frac{E_n(\omega)}{\log n} = \inf_{n\geq 1} \sup_{k\geq n} \frac{E_k(\omega)}{\log k} \quad \text{equals to}$$

(a) $\forall \epsilon > 0$,
$\omega \in \cup_{n\geq 1} \cap_{k\geq n} [\frac{E_k}{\log k} \leq 1 + \epsilon] = \liminf_{n\to\infty} [\frac{E_n}{\log n} \leq 1 + \epsilon]$.

(b) $\forall \epsilon > 0$, $\frac{E_n(\omega)}{\log n} > 1 - \epsilon$ for infinitely often; i.e.,
$\omega \in \limsup_{n\to\infty} [\frac{E_n}{\log n} > 1 - \epsilon]$.

Example 4.5.2 (Behavior of exponential random variables)
Proof continued

Then let $\epsilon_k \downarrow 0$,

$$[\limsup_{n\to\infty} \frac{E_n}{\log n} = 1] = \cap_k \left\{ \liminf_{n\to\infty} [\frac{E_n}{\log n} \leq 1 + \epsilon_k] \right\} \cap \cap_k \left\{ \limsup_{n\to\infty} [\frac{E_n}{\log n} > 1 - \epsilon_k] \right\}.$$

We note that $\sum_n P[\frac{E_n}{\log n} > 1 - \epsilon_k] = \sum_n \frac{1}{n^{1-\epsilon_k}} = \infty$. Thus $P\{\limsup_{n\to\infty}[\frac{E_n}{\log n} > 1 - \epsilon_k]\} = 1$. And $\sum_n P[\frac{E_n}{\log n} > 1 + \epsilon_k] = \sum_n \frac{1}{n^{1+\epsilon_k}} < \infty$, thus $P\{\limsup_{n\to\infty}[\frac{E_n}{\log n} > 1 + \epsilon_k]\} = 0$ implies $P\{\liminf_{n\to\infty}[\frac{E_n}{\log n} \leq 1 + \epsilon_k]\} = 1 - P\{\limsup_{n\to\infty}[\frac{E_n}{\log n} > 1 + \epsilon_k]\} = 1$. Therefore, $[\limsup_{n\to\infty} \frac{E_n}{\log n} = 1] = 1$.

4.5.3 Kolmogorov Zero-One Law

Let $\{X_n\}$ be a sequence of random variables and define

$$\mathcal{F}'_n = \sigma(X_{n+1}, X_{n+2}, \dots), \quad n = 1, 2, \dots$$

The **tail** σ-algebra $\mathcal{T}$ is defined as

$$\mathcal{T} = \cap_n \mathcal{F}'_n = \lim_{n\to\infty} \sigma(X_n, X_{n+1}, \dots).$$

If $A \in \mathcal{T}$, we will call A a **tail event** and similarly a random variable measurable with respect to $\mathcal{T}$ is called a **tail random variable**.

4.5.3 Kolmogorov Zero-One Law

Observe that

1. $$\{\omega : \sum_{n=1}^{\infty} X_n(\omega) \text{ converges}\} \in \mathcal{T}.$$

2. $$\limsup_{n\to\infty} X_n, \liminf_{n\to\infty} X_n, \{\omega : \lim_{n\to\infty} X_n(\omega) \text{ exists}\} \in \mathcal{T}$$

3. Let $S_n = X_1 + \cdots + X_n$. Then
$$\{\omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = 0\} \in \mathcal{T}.$$

4.5.3 Kolmogorov Zero-One Law

Call a σ-algebra, all of whose events have probability 0 or 1 **almost trivial**. One example is the σ-algebra $\{\emptyset, \Omega\}$.

Theorem 4.5.3 Kolmogorov Zero-One Law

If $\{X_n\}$ are independent random variables with tail σ-algebra $\mathcal{T}$, then $\Lambda \in \mathcal{T}$ implies $P(\Lambda) = 0$ or 1 so that the tail σ-albegra is almost trivial.

Corollary 4.5.1

Let $\{X_n\}$ be independent random variables. Then the followings are true.

(a) The event $[\sum_n X_n \text{ converges}]$ has probability 0 or 1.

(b) The random variables $\limsup_{n\to\infty} X_n$ and $\liminf_{n\to\infty} X_n$ are constant with probability 1.

(c) The event $\{\omega : S_n(\omega)/n \to 0\}$ has probability 0 or 1.

4.5.3 Kolmogorov Zero-One Law

Lemma 4.5.1 Almost trivial σ-algebras

Let $\mathcal{G}$ be an almost trivial σ-algebra and let X be a random variable measurable with respect to $\mathcal{G}$. Then there exists c such that $P[X = c] = 1$.

Proof: Let $F(x) = P[X \leq x]$. Then F is non-decreasing and since $[X \leq x] \in \sigma(X) \subset \mathcal{G}$, $F(x) = 0$ or 1 for each $x \in \mathbb{R}$. Let

$$c = \sup\{x : F(x) = 0\}.$$

The distribution function must have a jump of size 1 at c and thus

$$P[X = c] = 1.$$

Proof of the Komogorov Zero-One Law. Suppose $\Lambda \in \mathcal{T}$. we show Λ is independent of itself; i.e. $P(\Lambda \cap \Lambda) = P(\Lambda) = P(\Lambda)^2$. Thus $P(\Lambda) = 0$ or 1.

To show this, define $\mathcal{F}_n = \sigma(X_1, \dots, X_n) = \bigvee_{j=1}^{\infty} \sigma(X_j)$, so that $F_n \uparrow$ and $\mathcal{F}_\infty = \sigma(X_1, \dots,) = \bigvee_{j=1}^{\infty} \sigma(X_j) = \bigvee_{n=1}^{\infty} \mathcal{F}_n$. Note that

$$\Lambda \in \mathcal{T} \subset F_n' = \sigma(X_{n+1}, X_{n+2}, \dots) \subset \mathcal{F}_\infty.$$

Now for all n, $\Lambda \in \mathcal{F}_n'$. Since $\mathcal{F}_n \perp \mathcal{F}_n'$, we have $\Lambda \perp \mathcal{F}_n$ for all n, and therefore $\Lambda \perp \cup_n \mathcal{F}_n$.

Let $\mathcal{C}_1\{\Lambda\}$ and $\mathcal{C}_2 = \cup_n \mathcal{F}_n$. Then $\mathcal{C}_i$ is a π-system, $i = 1, 2$, $\mathcal{C}_1 \perp \mathcal{C}_2$ and therefore the Basic Criterion 4.1.1. implies $\sigma(\mathcal{C}_1) = \{\emptyset, \Omega, \Lambda, \Lambda^c\} \perp \sigma(\mathcal{C}_2) = \bigvee_n \mathcal{F}_n = \mathcal{F}_\infty$. We have $\Lambda \in \sigma(\mathcal{C}_1)$ and also $\Lambda \in \mathcal{F}_\infty$, thus $\Lambda \perp \Lambda$.

HW 4: Section 4.6, Q2, Q5-6, Q11-14, Q16-17, Q18-20, Q22