$$\underset{\sim}{X}_n = \begin{pmatrix} X_{1n} \\ \vdots \\ X_{pn} \end{pmatrix} \qquad \underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix}$$

" $X_{jn} \xrightarrow{d} X_j$ for $j=1,\cdots,p$ ". $\Longrightarrow$ " $\underset{\sim}{X}_n \xrightarrow{d} \underset{\sim}{X}$ " ? No

Counterexample $\quad p=2.$ if $\underset{\sim}{X}_n = \begin{pmatrix} Z \\ -Z \end{pmatrix}$ $\quad Z \sim N(0,1)$

$\underset{\sim}{X}$ = standard bivariate Normal

$$\underset{\sim}{X} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

then $\quad X_{1n} \xrightarrow{d} X_1 \quad , X_{2n} \xrightarrow{d} X_2$

but $\quad \begin{pmatrix} X_{1n} \\ X_{2n} \end{pmatrix} \overset{d}{\not\to} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ (why?)

$$X_{1n} + X_{2n} = Z + (-Z) = 0 \overset{d}{\not\to} X_1 + X_2$$

If $\quad X_{jn}$'s, $j=1,\cdots,p$ are mutually independent.

and if $\quad X_{jn} \xrightarrow{d} N(\mu_j, \sigma_j^2)$

Then $\quad \underset{\sim}{X}_n = \begin{pmatrix} X_{1n} \\ \vdots \\ X_{pn} \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_p^2 \end{pmatrix} \right)$

Cramer-Wald device.

$$\underset{\sim}{X}_n \xrightarrow{d} \underset{\sim}{X} \in R^p \iff \forall \underset{\sim}{t} \in R^p$$

$$\underset{\sim}{t}^T \underset{\sim}{X}_n \xrightarrow{d} \underset{\sim}{t}^T \underset{\sim}{X}$$

Example: $\quad \underset{\sim}{X}_i, \ i=1,\cdots,n. \ \underset{\sim}{\text{iid}} \ N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ where $\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix}$,

$E(\underset{\sim}{X}_i) = \underset{\sim}{\mu}$, $Cov(\underset{\sim}{X}_i) = \Sigma$

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12}, \cdots, & \sigma_{1p} \\ \sigma_{21} & \ddots & \\ \vdots & & \\ \sigma_{p1} & \cdots & \sigma_p^2 \end{pmatrix}$$

Then $\quad \forall \underset{\sim}{t} \in \mathbb{R}^p \quad E(\underset{\sim}{t}^T \underset{\sim}{X}_i) = \underset{\sim}{t}^T \underset{\sim}{\mu}, \quad Var(\underset{\sim}{t}^T \underset{\sim}{X}_i) = \underset{\sim}{t}^T \Sigma \underset{\sim}{t}$

$\underset{\sim}{t}^T \underset{\sim}{X}_i$'s $\overset{iid}{\sim} N(\underset{\sim}{t}^T \underset{\sim}{\mu}, \underset{\sim}{t}^T \Sigma \underset{\sim}{t})$

by CLT. $\quad \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \underset{\sim}{t}^T \underset{\sim}{X}_i - \underset{\sim}{t}^T \underset{\sim}{\mu}\right) \xrightarrow{d} N(\underset{\sim}{0}, \underset{\sim}{t}^T \Sigma \underset{\sim}{t})$

$\Rightarrow \quad \underset{\sim}{t}^T \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \underset{\sim}{X}_i - \underset{\sim}{\mu}\right) \xrightarrow{d} \underset{\sim}{t}^T \underset{\sim}{X}$ where $\underset{\sim}{X} \sim N(\underset{\sim}{0}, \Sigma)$

by C-W Device

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \underset{\sim}{X}_i - \underset{\sim}{\mu}\right) \xrightarrow{d} N(0, \Sigma)$$

Delta Method.

$\sqrt{n}(\underset{\sim}{X}_n - \underset{\sim}{\mu}) \xrightarrow{d} \underset{\sim}{X}$ where $\underset{\sim}{X}_n, \underset{\sim}{X} \in \mathbb{R}^p$

Let $\underset{\sim}{g} : \mathbb{R}^p \to \mathbb{R}^k = \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}$

Suppose $\dfrac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_p} \\ & \vdots & & \\ \frac{\partial g_k}{\partial x_1} & \cdots & \cdots & \frac{\partial g_k}{\partial x_p} \end{pmatrix}_{k \times p}$

is continuous at a neighborhood of $\underset{\sim}{\mu}$

$g(x_1, x_2, x_3) = \begin{pmatrix} \overbrace{x_1 + x_2}^{g_1} \\ \underbrace{x_2 + x_3}_{g_2} \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^2$

$g_1 = g_1(x_1, x_2, x_3) = x_1 + x_2$

$g_2(x_1, x_2, x_3) = x_2 + x_3$

Then $\quad \sqrt{n}\left(\underset{\sim}{g}(\underset{\sim}{X}_n) - \underset{\sim}{g}(\underset{\sim}{\mu})\right) \xrightarrow{d} \left[\left.\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}}\right|_{\underset{\sim}{x} = \underset{\sim}{\mu}}\right] \times \underset{\sim}{X}$

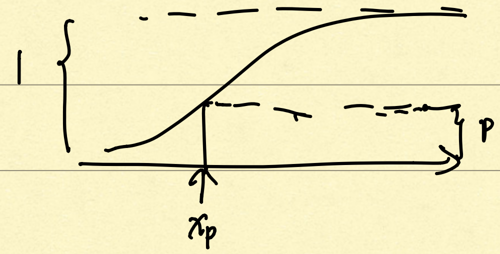
- Asymptotic Distribution of Sample Quantiles

Let $X_1, \cdots, X_n \overset{iid}{\sim} F$ where F is continous so that all observations are distinct with probability 1.

Order Statistic. $\quad X_{(1)} < \cdots < X_{(n)}$

$X_{(1:n)} < \cdots < X_{(n:n)}$

Population Quantile: for $0 < p < 1$. the p-th quantile is defined as $x_p = F^{-1}(p)$



Sample Quantile: $X_{(k:n)} \xrightarrow{\text{estimate}} x_p$ here $k = \lceil np \rceil$

ceiling of np

$k \neq n$
$k \neq 1$

Goal: $\sqrt{n}\left(X_{(k)} - x_p \right) \xrightarrow{d} ?$

or for $0 < p_1 < p_2 < 1$

$\lceil np_1 \rceil = k_1$ $\quad$ $k_2 = \lceil np_2 \rceil$

$$\sqrt{n}\begin{pmatrix} X_{(k_1:n)} - x_{p_1} \\ X_{(k_2:n)} - x_{p_2} \end{pmatrix} \xrightarrow{d} ?$$

- $X \sim F$, then $U = F(X) \sim U(0,1)$

$$X_{(1:n)} < \cdots < X_{(n:n)}$$

$$\underbrace{F(X_{(1:n)})}_{= U_{(1:n)}} < \cdots < \underbrace{F(X_{(n:n)})}_{= U_{(n:n)}}$$

Then $U_{(1:n)} < \cdots < U_{(n:n)}$ are the order statistics

from $U_1, \cdots, U_n \overset{iid}{\sim} U(0,1)$.

Lemma 1. (Question 4 in HW3) Let $Y_1 \cdots Y_{n+1} \overset{iid}{\sim} Exp(1)$

$$S_1 = Y_1$$

$$S_2 = Y_1 + Y_2$$

$$\vdots$$

$$S_n = Y_1 + Y_2 + \cdots + Y_n$$

$$S_{n+1} = Y_1 + Y_2 + \cdots + Y_n + Y_{n+1} \qquad (S_1 < S_2 < \cdots < S_n < S_{n+1})$$

given S_{n+1}, $\left(\frac{S_1}{S_{n+1}}, \frac{S_2}{S_{n+1}}, \cdots, \frac{S_n}{S_{n+1}} \right)$ has the same joint distribution as $\left(U_{(1:n)}, \cdots, U_{(n:n)}\right)$

by CLT: $\sqrt{k}\left(\frac{S_k}{k}-1\right) \xrightarrow{d} N(0,1)$ as $k\to+\infty$

If $n\to+\infty$, $\frac{k_1}{n}\to p_1$ then

$$\sqrt{n+1}\left(\frac{S_{k_1}}{n+1}-\frac{k_1}{n+1}\right) \leftarrow Y_1, \cdots, Y_{k_1}$$

$$= \underbrace{\sqrt{\frac{k_1}{n+1}}} \times \underbrace{\sqrt{k_1}\left(\frac{1}{k_1}S_{k_1}-1\right)} \xrightarrow{d} \sqrt{p_1}\, N(0,1) = N(0,p_1)$$

$k_1<k_2$

Similarly, if $\underbrace{n\to+\infty, \frac{k_1}{n}\to p_1 \text{ and } \frac{k_2}{n}\to p_2}$, then $\Rightarrow$ $k_2-k_1\to+\infty$

$$\sqrt{n+1}\left[\frac{1}{n+1}\underbrace{\left(S_{k_2}-S_{k_1}\right)} - \frac{k_2-k_1}{n+1}\right] \leftarrow Y_{k_1+1}, \cdots, Y_{k_2}$$

$$= \sqrt{\frac{k_2-k_1}{n+1}}\;\underbrace{\sqrt{k_2-k_1}\left[\frac{1}{k_2-k_1}\sum_{i=k_1+1}^{k_2}Y_i - 1\right]} \xrightarrow{d} \sqrt{p_2-p_1}\times \underline{N(0,1)} = N(0, p_2-p_1)$$

and similarly $\quad Y_{k_2+1}, \cdots, Y_{n+1}$

$$\sqrt{n+1}\left[\frac{1}{n+1}\left(S_{n+1}-S_{k_2}\right)-\frac{n+1-k_2}{n+1}\right]\xrightarrow{d} N(0, 1-p_2)$$

$$\sqrt{n+1}\begin{pmatrix}\frac{S_{k_1}}{n+1}-p_1\\ \frac{S_{k_2}-S_{k_1}}{n+1}-(p_2-p_1)\\ \frac{S_{n+1}-S_{k_2}}{n+1}-(1-p_2)\end{pmatrix}\xrightarrow{d} N\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}p_1&&\\&p_2-p_1&\\&&1-p_2\end{pmatrix}\right)$$

provided: $\sqrt{n}\left(\frac{k_1}{n}-p_1\right)\to 0$ and $\sqrt{n}\left(\frac{k_2}{n}-p_2\right)\to 0$.

Thm. If $U_{(1:n)} < \cdots < U_{(n:n)}$ are order statistices of an iid sample from $U(0,1)$ and if $n \to \infty$, $k_1 \to +\infty$, $k_2 \to +\infty$ in such a way that

$$\sqrt{n}\left(\frac{k_1}{n} - P_1\right) \to 0 \quad , \sqrt{n}\left(\frac{k_2}{n} - P_2\right) \to 0$$

where $0 < P_1 < P_2 < 1$

Then

$$\sqrt{n}\begin{pmatrix} \overset{F(X_{(k_1:n)})}{\overset{\|}{U_{(k_1:n)}}} - \overset{F(X_{p_1})}{\overset{\|}{P_1}} \\ \underset{F(X_{(k_2:n)})}{\underset{\|}{U_{(k_2:n)}}} - \underset{F(X_{p_2})}{\underset{\|}{P_2}} \end{pmatrix} \xrightarrow{d} N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{pmatrix} P_1(1-P_1) & P_1(1-P_2) \\ P_1(1-P_2) & P_2(1-P_2) \end{pmatrix}\right)$$

Proof:

$$\begin{pmatrix} U_{(k_1:n)} \\ U_{(k_2:n)} \end{pmatrix} \overset{d}{=} \begin{pmatrix} \frac{S_{k_1}}{S_{n+1}} \\ \frac{S_{k_2}}{S_{n+1}} \end{pmatrix} \quad \text{from Lemma 1.}$$

$$g(X_1, X_2, X_3) = \begin{pmatrix} \frac{X_1}{X_1 + X_2 + X_3} \\ \frac{X_1 + X_2}{X_1 + X_2 + X_3} \end{pmatrix}$$

$$g\left(\frac{S_{k_1}}{n+1}, \frac{S_{k_2} - S_{k_1}}{n+1}, \frac{S_{n+1} - S_{k_2}}{n+1}\right) = \begin{pmatrix} \frac{S_{k_1}}{S_{n+1}} \\ \frac{S_{k_2}}{S_{n+1}} \end{pmatrix} \quad \#$$

Corollary. If $X_{(1:n)} < \cdots < X_{(n:n)}$ are order statistics of a sample of size n from a distribution F having $f(x)$ continuous and positive in a neighborhood of the quantiles $\underset{X_{(k_1:n)}}{\underset{\uparrow}{x_{p_1}}}$ and $\underset{X_{(k_2:n)}}{\underset{\uparrow}{x_{p_2}}}$ with $P_1 < P_2$ where $k_1 = \lceil nP_1 \rceil$, $k_2 = \lceil nP_2 \rceil$

$$\sqrt{n}\begin{pmatrix} X_{(k_1:n)} - x_{p_1} \\ X_{(k_2:n)} - x_{p_2} \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{p_1(1-p_1)}{f(x_{p_1})^2} & \frac{p_1(1-p_2)}{f(x_{p_1})f(x_{p_2})} \\ \frac{p_1(1-p_2)}{f(x_{p_1})f(x_{p_2})} & \frac{p_2(1-p_2)}{f(x_{p_2})^2} \end{pmatrix} \right)$$

Proof.

$$g(y_1, y_2) = \begin{pmatrix} F^{-1}(y_1) \\ F^{-1}(y_2) \end{pmatrix} = \begin{pmatrix} g_1(y_1, y_2) \\ g_2(y_1, y_2) \end{pmatrix}$$

$$\frac{\partial g}{\partial \underset{\sim}{y}} = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial F^{-1}(y_1)}{\partial y_1}, & 0 \\ 0 & \frac{\partial F^{-1}(y_2)}{\partial y_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{f(F^{-1}(y_1))}, & \\ & \frac{1}{f(F^{-1}(y_2))} \end{pmatrix}$$

known $$\sqrt{n}\begin{pmatrix} F(X_{(k_1:n)}) - F(X_{p_1}) \\ F(X_{(k_2:n)}) - F(X_{p_2}) \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} p_1(1-p_1) & p_1(1-p_2) \\ p_1(1-p_2) & p_2(1-p_2) \end{pmatrix} \right)$$

$$\begin{pmatrix} X_{(k_1:n)} \\ X_{(k_2:n)} \end{pmatrix} = g\left(F(X_{(k_1:n)}), F(X_{(k_2:n)}) \right)$$

by Delta Method.

$$\sqrt{n}\begin{pmatrix} X_{(k_1:n)} - X_{p_1} \\ X_{(k_2:n)} - X_{p_2} \end{pmatrix} = \sqrt{n}\left(g\left(F(X_{(k_1:n)}), F(X_{(k_2:n)})\right) - g\left(F(X_{p_1}), F(X_{p_2})\right) \right)$$

$$\xrightarrow{d} \underbrace{\left[\frac{\partial g}{\partial \underset{\sim}{y}} \Bigg|_{\begin{pmatrix} F(X_{p_1}) \\ F(X_{p_2}) \end{pmatrix}} \right]}_{= \begin{pmatrix} \frac{1}{f(X_{p_1})} & \\ & \frac{1}{f(X_{p_2})} \end{pmatrix}} \times N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} p_1(1-p_1) & p_1(1-p_2) \\ p_1(1-p_2) & p_2(1-p_2) \end{pmatrix} \right)$$

$$\times \left[\frac{\partial g}{\partial \underset{\sim}{y}} \Bigg|_{\begin{pmatrix} F(X_{p_1}) \\ F(X_{p_2}) \end{pmatrix}} \right]^T \quad \begin{pmatrix} \frac{1}{f(X_{p_1})} & \\ & \frac{1}{f(X_{p_2})} \end{pmatrix}$$

$y = F(x)$ $\quad$ $x = F^{-1}(y)$ $\quad$ $F(x) =$ $F(F^{-1}(y)) = y$

$$f(F^{-1}(y)) \times \frac{\partial F^{-1}(y)}{\partial y} = 1$$

$$\frac{\partial F^{-1}(y)}{\partial y} = \frac{1}{f(F^{-1}(y))} \Longleftarrow$$

$$\frac{1}{f(F^{-1}(F(x_{p_1})))} = \frac{1}{f(x_{p_1})} \qquad \frac{1}{f(F^{-1}(F(x_{p_2})))} = \frac{1}{f(x_{p_2})} \quad \#$$

Example: Let M_n represent the median of a sample of size n. from $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$$

$$f(\mu) = \frac{1}{\sqrt{2\pi\sigma^2}}$$

$$\sqrt{n}(M_n - \mu) \xrightarrow{d} N\left(0, \frac{1}{4f^2(\mu)}\right) \qquad p = \frac{1}{2}$$

$$= N\left(0, \frac{\pi\sigma^2}{2}\right)$$

Example: Cauchy(μ, σ^2)

$$f(x) = \frac{1}{\pi\sigma} \times \frac{1}{1 + \left[\frac{(x-\mu)}{\sigma}\right]^2}$$

$$\sqrt{n}(M_n - \mu) \xrightarrow{d} N\left(0, \frac{\pi^2\sigma^2}{4}\right) \qquad \frac{\frac{1}{2}(1-\frac{1}{2})}{f^2(\mu)}$$

$$\overline{X}_n \not\xrightarrow{P} \mu$$

x_p: pth quantile

$$\boxed{\sigma} = \frac{IQR}{2} = \frac{x_{\frac{3}{4}} - x_{\frac{1}{4}}}{2}$$ semi-interquantile range

$$\hat{\sigma} = \frac{X_{(\lceil \frac{3n}{4} \rceil : n)} - X_{(\lceil \frac{n}{4} \rceil : n)}}{4}$$

$x_{\frac{3}{4}} = \mu + \sigma$

$x_{\frac{1}{4}} = \mu - \sigma$

$$\text{If. } \sqrt{n}\begin{pmatrix} X_{(\lceil \frac{3n}{4} \rceil : n)} - x_{\frac{3}{4}} \\ X_{(\lceil \frac{n}{4} \rceil : n)} - x_{\frac{1}{4}} \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \pi^2\sigma^2 \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}\right)$$

take, $g(x_1, x_2) = \frac{x_1 - x_2}{2}$ apply delta method

$$\text{then } \sqrt{n}(\hat{\sigma} - \sigma) = \sqrt{n}\left(\frac{X_{(\lceil \frac{3n}{4} \rceil : n)} - X_{(\lceil \frac{n}{4} \rceil : n)}}{2} - \frac{x_{\frac{3}{4}} - x_{\frac{1}{4}}}{2}\right)$$

$$\xrightarrow{d} N(0, ???)$$

$\sqrt{n}(\underset{\sim}{X}_n - \underset{\sim}{X}) \xrightarrow{d} N(\mu, \Sigma)$

$\sqrt{n}(t^T \underset{\sim}{X}_n - t^T \underset{\sim}{X}) \xrightarrow{d} N(t^T\mu, t^T\Sigma t)$

$$\left(\frac{1}{2}, -\frac{1}{2}\right) \times \pi^2 \sigma^2 \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \frac{\pi^2 \sigma^2}{4}$$

$$\sqrt{n}(\hat{\sigma} - \sigma) \xrightarrow{d} N\left(0, \frac{\pi^2 \sigma^2}{4}\right)$$

Asymptotic Theory of Extreme Order Statistics

- Example $X_1 \cdots X_n \overset{iid}{\sim} U(0, \theta)$ $\quad \hat{\theta} = \max(X_i) = X_{(n:n)}$

Find a_n, b_n. such that $\frac{X_{(n)} - a_n}{b_n} \xrightarrow{d}$?

$a_n = \theta \quad b_n = ?$

$$P(\hat{\theta} \le t) = P(X_{(n:n)} \le t)$$
$$= \prod_{i=1}^{n} P(X_i \le t)$$
$$= \left(\frac{t}{\theta}\right)^n$$

$$P\left(\frac{\hat{\theta} - \theta}{b_n} \le t\right) \to ?$$

$$= P(\hat{\theta} \le b_n t + \theta) = \left(\frac{b_n t + \theta}{\theta}\right)^n$$
$$= \left(\frac{b_n t}{\theta} + 1\right)^n$$

If $b_n = \frac{1}{n}$ $\quad = \left(\frac{t/\theta}{n} + 1\right)^n \to e^{\frac{t}{\theta}}$

$$P(n(\hat{\theta} - \theta) \le t) \to e^{\frac{t}{\theta}} \quad \text{as} \quad n \to +\infty$$

$X_1, \cdots, X_n \overset{iid}{\sim} F$ continous

$$M_n = \max_{1 \le i \le n} X_i$$

Goal: to find $\underbrace{a_n, b_n}_{\text{real sequence}}$ such that. $\frac{M_n - a_n}{b_n} \xrightarrow{d}$ a non-degenerated distr.

$$P\left(\frac{M_n - a_n}{b_n} \le x \right) = P(M_n \le a_n + b_n x)$$

$$= P(X_1, \cdots, X_n \le a_n + b_n x)$$

$$= \prod_{i=1}^{n} P(X_i \le a_n + b_n x)$$

$$= F(a_n + b_n x)^n$$

$$\left(1 - \frac{x}{n}\right)^n \to e^{-x} \text{ !!!}$$

Goal find a_n, b_n s.t. $F(a_n + b_n x)^n \to G(x)$ as $n \to +\infty$

Definition (slowly varying) a function $c: [0, \infty) \to \mathbb{R}$ is slowly varying

if for every $x > 0$, $\frac{c(tx)}{c(t)} \to 1$ as $t \to +\infty$

Example. • $c(t) = \log t$ $\quad C(tx) = \log(tx) = \log t + \log x$

$$\frac{C(tx)}{c(t)} = \frac{\log t + \log x}{\log t} \to 1 \quad \text{as } t \to +\infty$$

• $C(t) = (\log t)^{\gamma}$ $\quad \frac{C(tx)}{C(t)} = \left(\frac{\log t + \log x}{\log t} \right)^{\gamma} = (\to 1)^{\gamma} \to 1$

• $C(t) = t^{\gamma}$ $\quad \gamma > 0$ $\quad \frac{C(tx)}{C(t)} = \frac{(tx)^{\gamma}}{t^{\gamma}} = x^{\gamma} \to 1$ as $t \to +\infty$?

Thm: Let $F(x)$ denote the distribution function of a r.v. X.

Let x_0 denote the upper boundary, (possibly $+\infty$), of the distribution of X:

$$x_0 = \sup\{x : F(x) < 1\}$$

Example: $X \sim Exp(1)$, $x_0 = +\infty$

$X \sim N(\mu, \sigma^2)$, $x_0 = +\infty$

$X \sim Beta(\ ,\ )$, $x_0 = 1$

$X \sim U(0, \theta)$, $x_0 = \theta$

(a) If $x_0 = +\infty$ and $1 - F(x) = x^{-\gamma} c(x)$ for some $\gamma > 0$, and some slowly varying $c(x)$.

then $F(b_n x)^n \to G_{1,\gamma}(x) = \begin{cases} \exp(-x^{\gamma}) & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$

where b_n is such that $1 - F(b_n) = \frac{1}{n}$.

$$\frac{M_n - 0}{b_n} \xrightarrow{d} G_{1,\gamma}$$

Proof of (a). we observed that $b_n \to +\infty$ as $n \to +\infty$

$$F(b_n x)^n = \left[1 - (b_n x)^{-\gamma} c(b_n x)\right]^n$$

$$= \left[1 - \frac{n\, c(b_n x)}{n (b_n x)^{\gamma}}\right]^n$$

$$= \left[1 - \frac{x^{-\gamma}}{n} \times \frac{n\, c(b_n x)}{b_n^{\gamma}}\right]^n$$

If $\frac{n\, c(b_n x)}{b_n^{\gamma}} \to 1$ as $n \to +\infty$ we have $F(b_n x)^n \to e^{-x^{-\gamma}}$

known $1 - F(b_n) = \frac{1}{n}$ and $1 - F(x) = x^{-\gamma} c(x)$

thus, $\frac{1}{n} = b_n^{-\gamma} c(b_n)$

$$1 = \frac{n\, c(b_n)}{b_n^{\gamma}} \quad \text{or} \quad b_n^{\gamma} = n\, c(b_n)$$

Consequently. $\frac{n\, c(b_n x)}{b_n^{\gamma}} = \frac{n\, c(b_n x)}{n\, c(b_n)} = \frac{c(b_n x)}{c(b_n)} \to 1$ as $b_n \to +\infty$

Example: t_v $\quad f(x) = \frac{\text{Constant}}{(v + x^2)^{\frac{v+1}{2}}} \sim c\, x^{-(v+1)}$ $\quad \frac{c}{v} x^{-v}$

$$1 - F(x) = x^{-v} c(x) \quad \text{for some function } c(x) \to \frac{c}{v}.$$

$$1 - F(b_n) = \frac{1}{n}$$

$$1 - F(b_n) \sim c(b_n)\, b_n^{-v} = \frac{c}{v} \times \frac{1}{b_n^{v}} = \frac{1}{n}$$

$$\Rightarrow \quad b_n = \left(\frac{cn}{v}\right)^{\frac{1}{v}}$$

$$\frac{M_n}{b_n} = \frac{M_n}{\left(\frac{cn}{v}\right)^{\frac{1}{v}}} \xrightarrow{d} G_{1,v}$$

If Cauchy, $v = 1$, $c = \frac{1}{\pi}$ $\quad \frac{\pi M_n}{n} \xrightarrow{d} G_{1,1} =$ Standard Exponential

1

(b) If $x_0 < +\infty$ and $1 - F(x) = (x_0 - x)^{\gamma}\, c\left(\frac{1}{x_0 - x}\right)$ for some $\gamma > 0$

and some slowly varying $c(x)$, then

$$F(x_0 + b_n x)^n \rightarrow G_{2,\gamma} = \begin{cases} \exp\{-(-x)^{\gamma}\} & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

where $1 - F(x_0 - b_n) = \frac{1}{n}$ $\qquad \frac{M_n - x_0}{b_n} \xrightarrow{d} G_{2,\gamma}$

$\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$

Proof. for $x < 0$

$$F(x_0 + b_n x)^n = \left[1 - (x_0 - (x_0 + b_n x))^{\gamma}\, c\left(\frac{1}{x_0 - (x_0 + b_n x)}\right)\right]^n$$

$$= \left[1 - (-b_n x)^{\gamma}\, c\left(\frac{1}{-b_n x}\right)\right]^n$$

$$= \left[1 - \frac{(-x)^{\gamma}}{n} \times \underbrace{n\,(b_n)^{\gamma} \times c\left(\frac{1}{-b_n x}\right)}\right]^n$$

It suffices to show $\quad n(-b_n)^{\gamma}\, c\left(\frac{1}{-b_n x}\right) \rightarrow 1$ as $n \to +\infty$

what is the limit of b_n? known from $1 - F(x_0 - b_n) = \frac{1}{n}$

$b_n \to 0$ as $n \to +\infty$

$\frac{1}{b_n} \to +\infty$

$$\frac{c\left(\frac{1}{-b_n x}\right)}{c\left(\frac{1}{b_n}\right)} = \frac{c\left(\frac{1}{b_n} \times \frac{1}{-x}\right)}{c\left(\frac{1}{b_n}\right)} \rightarrow 1 \quad \text{as } n \to +\infty$$

slowly varying

$$\lim_{n\to\infty} n(+b_n)^{\gamma} c\left(\frac{1}{-b_n x}\right) = \lim_{n\to\infty} n(+b_n)^{\gamma} c\left(\frac{1}{b_n}\right) = 1$$

$$\frac{1}{n} = 1 - F(x_0 - b_n) = (x_0 - (x_0 - b_n))^{\gamma} c\left(\frac{1}{x_0 - (x_0 - b_n)}\right)$$

$$= b_n^{\gamma} c\left(\frac{1}{b_n}\right)$$

Example. $X \sim \text{Beta}(\alpha, \beta)$

$$f(x) = c \; x^{\alpha-1} (1-x)^{\beta-1} I(0 < x < 1)$$

$$c = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$\frac{M_n - a_n}{b_n} \xrightarrow{d} ?$$

In this case $x_0 = 1$ $\quad a_n = 1$

as $x \nearrow 1$ $\quad f(x) \sim c(1-x)^{\beta-1}$

$$1 - F(x) \sim c \int_x^1 (1-u)^{\beta-1} du = \frac{c(1-x)^{\beta}}{\beta}$$

Here, we can take. $\gamma = \beta$, $x_0 = 1$

$$\frac{1}{n} = 1 - F(x_0 - b_n) = 1 - F(1 - b_n) = \frac{c(1-(1-b_n))^{\beta}}{\beta} = \frac{c b_n^{\beta}}{\beta}$$

$$\Rightarrow b_n^{\beta} \sim \frac{\beta}{nc} \qquad \Gamma(\beta+1) = \Gamma(\beta) \times \beta$$

we can take $\quad b_n = \left(\frac{\Gamma(\alpha)\Gamma(\beta+1)}{n\Gamma(\alpha+\beta)}\right)^{\frac{1}{\beta}}$

$$\frac{M_n - 1}{b_n} \xrightarrow{d} G_{2,\beta}$$

If X_i's $\sim U(0,1)$ $\quad x_0 = 1$, $b_n = \frac{1}{n}$

$$n(M_n - 1) \xrightarrow{d} G_{2,1} = -G_{1,1} = -\text{standard Exponential!}$$

(c) If there exists a function $R(t)$ such that for all x.

$$P(X>t+xR(t)\mid X>t)=\frac{1-F(t+xR(t))}{1-F(t)}=\frac{S(t+xR(t))}{S(t)}\rightarrow e^{-x}$$ as $t\to x_0$

(could be finite or $+\infty$)

then. Let $1-F(a_n)=\frac{1}{n}$. $b_n=R(a_n)$

we have $\frac{M_n-a_n}{b_n}\xrightarrow{d}G_{3,r}$ where $G_{3,r}(x)=\exp\{-e^{-x}\}$

Proof. $P\left(\frac{M_n-a_n}{b_n}\le x\right)=P(M_n\le a_n+b_nx)=\prod_{i=1}^{n}P(X_i\le a_n+b_nx)$

$$=F(a_n+b_nx)^n$$

$(1-\frac{x}{n})^n\to e^{-x}$

$$=(1-\{1-F(a_n+b_nx)\})^n$$

$$=\left(1-\frac{n(1-F(a_n+b_nx))}{n}\right)^n\xrightarrow{\text{Goal}}\exp\{-e^{-x}\}$$

it suffices to show $\underline{n(1-F(a_n+b_nx))}\to e^{-x}$ as $n\to+\infty$

$$n(1-F(a_n+b_nx))=\frac{1-F(a_n+R(a_n)x)}{\frac{1}{n}}=\frac{1-F(a_n+R(a_n)x)}{1-F(a_n)}\quad(*)$$

by the definition of a_n, we know $a_n\to x_0$ as $n\to+\infty$

Thus $(*)\to e^{-x}$ as $n\to+\infty$ #

Note: It is a remarkable fact that the converse to this theorem is true.

If for some normalizing sequences a_n and b_n

Such that $\frac{M_n-a_n}{b_n}\xrightarrow{d}G$ a non-degenerate distribution (up to change of location and scale)

then G is one of the three type:

① $G=G_{1,r}$. for $r>0$

② $G=G_{2,r}$. for $r>0$

③ G_3

Example: $F(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{u^2}{2}\right\} du$ $\quad N(0,1)$

Lemma: $\sqrt{2\pi}\,(1-\Phi(x)) = \int_{x}^{+\infty} \exp\left\{-\frac{u^2}{2}\right\} du \sim \frac{1}{x}\exp\left\{-\frac{x^2}{2}\right\}$ as $x \to +\infty$

Proof. $\lim_{x\to+\infty} \frac{\sqrt{2\pi}\,(1-\Phi(x))}{\frac{1}{x}\exp\left\{-\frac{x^2}{2}\right\}} = 1$ is the goal

Take derivative

L'Hospital's rule

$$= \lim_{x\to+\infty} \frac{-\sqrt{2\pi}\,\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}}{-\frac{1}{x^2}\exp\left\{-\frac{x^2}{2}\right\} + \frac{1}{x}\exp\left\{-\frac{x^2}{2}\right\}(-x)}$$

$$= \lim_{x\to+\infty} \frac{-1}{-\frac{1}{x^2}-1} = \lim_{x\to+\infty} \frac{x^2}{x^2+1} = 1$$

Back to $N(0,1)$. focus on type (c).

find $R(t)$: $\frac{1-\Phi(t+xR(t))}{1-\Phi(t)} \longrightarrow e^{-x}$ as $t\to+\infty$

based on the lemma. $\frac{1-\Phi(t+xR(t))}{1-\Phi(t)} \sim \frac{\frac{1}{(t+xR(t))} \times \exp\left\{-\frac{(t+xR(t))^2}{2}\right\}}{\frac{1}{t}\exp\left\{-\frac{t^2}{2}\right\}}$

$$= \frac{1}{t+xR(t)} \times \exp\left\{-\frac{\cancel{t^2}+2xR(t)t+x^2R^2(t)}{2}\right\} \times \frac{t}{\exp\left\{-\cancel{\frac{t^2}{2}}\right\}}$$

$$= \frac{t}{t+xR(t)} \exp\left\{-txR(t) - \frac{x^2R^2(t)}{2}\right\} \longrightarrow e^{-x} \text{ as } t\to+\infty \quad (*)$$

we see that $R(t) = \frac{1}{t}$ make $(*)$ hold.

Thus $\dfrac{M_n - a_n}{b_n} = \dfrac{M_n - a_n}{R(a_n)} = a_n(M_n - a_n) \xrightarrow{d} G_3$

where $1 - F(a_n) = \frac{1}{n}$ $\quad a_n = F^{-1}(1-\frac{1}{n}) = Z_{1-\frac{1}{n}}$ $\quad Z_\alpha$

In fact $\quad a_n \approx \sqrt{2\log n}$

Motivating Example: ① $X_1, \cdots, X_n \sim \text{Unif}(-\theta, \theta)$ How to estimate θ?

$\hat{\theta} = \max(|X_{(n,n)}|, |X_{(1:n)}|)$

$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{?} ?$

$n(\hat{\theta} - \theta) \xrightarrow{?} ?$

Can we find a_n, b_n such that $\frac{\hat{\theta} - a_n}{b_n} \xrightarrow{d} G_1$?

② $X_1, \cdots, X_n \sim \text{unif}(\theta - 0.5, \theta + 0.5)$ How to est θ?

$\hat{\theta}_1 = X_{(n:n)} - 0.5$, $\hat{\theta}_2 = X_{(1:n)} + 0.5$

$\hat{\theta}_3 = \bar{X}_n$ $\qquad \hat{\theta}_4 = \text{median}(X_i\text{'s})$

$\hat{\theta}_5 = \dfrac{X_{(n:n)} + X_{(1:n)}}{2}$. $\qquad \hat{\theta}_6 = \dfrac{X_{(k:n)} + X_{(n+1-k:n)}}{2}$

③ How to estimation population Range?

$\hat{R}_n = X_{(n:n)} - X_{(1:n)}$

$\hat{R}_n = 2(Q_3 - Q_1)$

- Asymptotic Joint Distributions of Extrema

Suppose $X_1, \cdots, X_n \overset{iid}{\sim} \text{Uniform}(0,1)$

for any fixed k $\qquad \left(\frac{k}{n} \to 0 \text{ as } n \to +\infty \Leftrightarrow \underset{\text{quartile}}{p = 0}\right)$

(a) $\qquad n\left(\underline{X_{(1:n)}, X_{(2:n)}, \cdots, X_{(k:n)}}\right) \xrightarrow{d} (S_1, S_2, \cdots, S_k)$

where $S_j = \sum_{i=1}^{j} Y_i$ and the Y_i's are iid $\text{Exp}(1)$

(b) for fixed $0 < p_1 < p_2 < \cdots < p_n < 1$. the three vectors

$$n\left(X_{(1:n)}, \cdots, X_{(k:n)} \right) \quad ①$$

joint quantiles $\longrightarrow$ $\sqrt{n}\left(X_{(np_1:n)} - p_1, \; \cdots, \; X_{(np_n:n)} - p_n \right) \quad ②$

$$n\left(1 - X_{(n:n)}, \cdots, 1 - X_{(n-k+1,n)} \right) \quad ③$$

are asymptotically independent, with distribution of ① and ③ as in (a), of ② as in the quantile section.

$$② \xrightarrow{d} N\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{pmatrix} p_1(1-p_1) & p_1(1-p_2) & p_1(1-p_3) \cdots & p_1(1-p_n) \\ & p_2(1-p_2) & p_2(1-p_3) \cdots & p_2(1-p_n) \\ * & & \ddots & \\ & & & p_n(1-p_n) \end{pmatrix} \right)$$