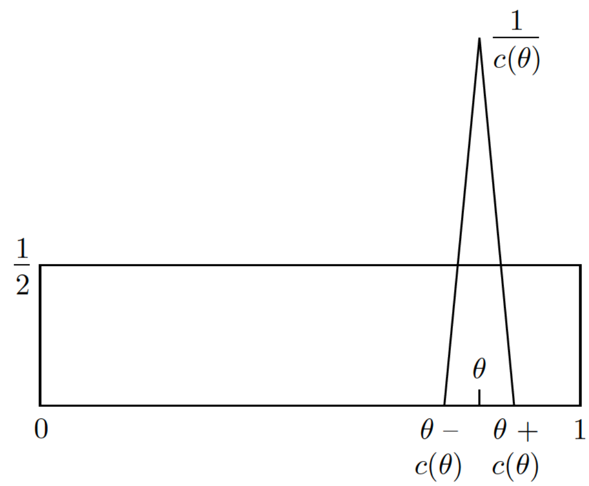
STAT 823, Homework 1, due Jan. 23, 2020

1. Let $X_1, \ldots, X_n$ be i.i.d. with a distribution which with probability θ is the uniform distribution on $(-1, 1)$, and with probability $(1-\theta)$ is equal to the triangular distribution with density

$$\frac{1}{c(\theta)}\left[1 - \frac{|x-\theta|}{c(\theta)}\right], \quad \theta - c(\theta) < x < \theta + c(\theta), \tag{1}$$

where $c(\theta)$ is a continuous decreasing function of θ with $c(0) = 1$ and $0 \le c(\theta) \le 1-\theta$ for $0 \le \theta < 1$, and with $c(\theta) \to 0$ as $\theta \to 1$.

Below shows both the uniform density and triangular density corresponding to a value of θ close to 1.



1. Show that (1) defines a probability density.
2. Give an example of a function $c(\theta)$ satisfying the conditions stated following (1).
3. Prove that the MLE $\hat{\theta}_n$ exists for all n but instead of being consistent, i.e., of converging in probability to θ, it converges to 1 as $n \to \infty$ no matter what the true value of θ. (Hint: Ferguson, 1982, An inconsistent maximum likelihood estimate, *JASA*).