1. Let $X_1, \ldots, X_n$ be a sequence of random variables. Denote by $F_i(x)$ the cdf function of $X_i$ for each $i$. Assume that $\lim_{n \to \infty} F_n(x) = F(x)$ for some function $F$. Prove or provide a counterexample to each of the following statements:

(a) $0 \leq F(x) \leq 1$

(b) For any $x \leq x'$, $F(x) \leq F(x')$

(c) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

(d) $F$ is right continuous.

2. Let $X_1, \ldots, X_n$ be iid from a distribution with density

$$f(x) = \frac{1}{\sqrt{2\pi x^3}} \exp \left\{ -\frac{1}{2x} \right\}, \quad x > 0$$

Show that for any $n$, the sample mean $\bar{X}_n$ has the same distribution as $nX_1$; i.e., the sample mean is much more variable than the single observation $X_1$.

3. Let $S_n$ be the Binomial distribution $B(n, p_n)$.

(a) Show that when $p_n = p$,

$$\frac{S_n - np}{\sqrt{npq}} \xrightarrow{d} N(0, 1)$$

where $q = 1 - p$.

(b) Show that when $p_n = \lambda/n$, $S_n$ converges in probability to a Poisson distribution.

4. In the standard two-way random effect model, it is assumed that

$$X_{ij} = A_i + U_{ij} \quad \text{for } j = 1, \ldots, m, i = 1, \ldots, s,$$

with the $A$’s and $U$’s all independently, normally distributed with

$$E(A_i) = \xi, Var(A_i) = \sigma_A^2, E(U_{ij}) = 0, Var(U_{ij}) = \sigma^2.$$ 

Show that $\bar{X} = \sum_{i=1}^s \sum_{j=1}^m X_{ij}/(sm)$ is

(a) not consistent for $\xi$ if $m \to \infty$ and $s$ remains fixed,

(b) consistent for $\xi$ if $s \to \infty$ and $m$ remains fixed.

5. Prove: if $U_{1n} = o_p(V_{1n})$ and $U_{2n} = O_p(V_{2n})$, then $U_{1n}U_{2n} = o_p(V_{1n}V_{2n})$. 

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