STAT 823, Homework 2, due Feb. 6, 2020

1. Let $X_1, \ldots, X_n$ be a sequence of random variables. Denote by $F_i(x)$ the cdf function of X_i for each i. Assume that $\lim_{n\to\infty} F_n(x) = F(x)$ for some function F. Prove or provide a counterexample to each of the following statements:

(a) $0 \le F(x) \le 1$

(b) For any $x \le x'$, $F(x) \le F(x')$

(c) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$

(d) F is right continuous.

2. Let $X_1, \ldots, X_n$ be iid from a distribution with density

$$f(x) = \frac{1}{\sqrt{2\pi x^3}} \exp\left\{-\frac{1}{2x}\right\}, \quad x > 0$$

Show that for any n, the sample mean $\bar{X}_n$ has the same distribution as nX_1; i.e., the sample mean is much more variable than the single observation X_1.
3. Let S_n be the Bionmial distribution $B(n, p_n)$.

(a) Show that when $p_n = p$,

$$\frac{S_n - np}{\sqrt{npq}} \xrightarrow{d} N(0,1)$$

where $q = 1 - p$.

(b) Show that when $p_n = \lambda/n$, S_n converges in probability to a Poisson distribution.

4. In the standard two-way random effect model, it is assumed that

$$X_{ij} = A_i + U_{ij} \quad \text{for } j = 1, \ldots, m, i = 1, \ldots, s,$$

with the A's and U's all independently, normally distributed with

$$E(A_i) = \xi, Var(A_i) = \sigma_A^2, E(U_{ij}) = 0, Var(U_{ij}) = \sigma^2.$$

Show that $\bar{X} = \sum_{i=1}^{s} \sum_{j=1}^{m} X_{ij}/(sm)$ is

(a) not consistent for ξ if $m \to \infty$ and s remains fixed,

(b) consistent for ξ if $s \to \infty$ and m remains fixed.

5. Prove: if $U_{1n} = o_p(V_{1n})$ and $U_{2n} = O_p(V_{2n})$, then $U_{1n}U_{2n} = o_p(V_{1n}V_{2n})$.