STAT 823, Homework 4, due Mar. 10, 2020

1. Let $X_1, X_2, \ldots,$ be a sample from Uniform$(0, 2\mu)$.

(a) Find the asymptotic distribution of the median.

(b) Find the asymptotic distribution of the midquartile range; i.e., $\{X_{(\lceil 3n/4 \rceil:n)} + X_{(\lceil n/4 \rceil:n)}\}/2$.

(c) Find the asymptotic distribution of $2X_{(\lceil 3n/4 \rceil:n)}/3$.

(d) Compare these three estimates of μ.

2. Let $X_1, X_2, \ldots$ be a sample from the beta distribution with density $f(x|\theta) = \theta x^{\theta-1} I(0 < x < 1)$, where $\theta > 0$,

(a) Let M_n denote the sample median and $m(\theta)$ the population median as a function of θ. What is the asymptotic distribution of $\sqrt{n}\{M_n - m(\theta)\}$.

(b) Let $\hat{\theta}_n = \log(1/2)/\log(M_n)$. Show $\hat{\theta}_n$ converges to θ in probability.

(c) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$.

3. For the following distributions, find the normalization such that $(M_n - a_n)/b_n$ has a nondegenerate limit if any exists (M_n is the large order statistics of a sample of size n).

(a) $f(x) = e^x I(x < 0)$.

(b) $f(x) = (2/x^3) I(x > 1)$.

(c) $F(x) = 1 - \exp\{-x/(1-x)\}$ for $0 < x < 1$.

4. Let $X_1, \ldots, X_n$ be a sample from Uniform$(\theta - 0.5, \theta + 0.5)$. Among the various estimates of θ, one may use the sample median $\hat{\theta}_1$, and one may use the midrange, $\hat{\theta}_2 = (\max X_i + \min X_i)/2$. Compare the 95% confidence interval for θ obtained from these two estimates, when $n = 100$.